A GEOMETRICAL APPROACH IN STRING FIELD THEORY

By

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To My Parents

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TABLE OF CONTENTS

page

ACKNOWLEDGEMENTS iii

ABSTRACT v

INTRODUCTION 1

GEOMETRIC QUANTIZATION 4

STRING THEORY AS A KÄHLER GEOMETRY OF LOOP SPACE 11

- Why Loop Space? 11
- The Idea of Bowick and Rajeev's Proposal 12
- The Loop Space of A Riemann Manifold is A Kähler Manifold 14

SUPERSTRING 21

- Geometric Quantization of The Superstring 21
- Curvature of the Canonical Line Bundle 28

THE CONFORMAL FACTOR AND BRST FORMALISM 34

APPENDIX Q-BALLS IN SUPERFLUID ^3He 44

REFERENCES 57

BIOGRAPHICAL SKETCH 61

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String theory is believed to be a consistent quantum theory which unifies gravity and other forces and matter fields. Since, according to Einstein, gravity tells us the geometry of space-time, string theory should be formulated in a way that is independenet of the background geometry. An attempt is made to analyze one of the proposals on geometric formulation of string theory.

Geometric quantization is explained in detail. The loop space of background geometry is studied also to see whether we get consistent string theories. An extension to superstring is given, showing that the phase space of open superstring is also a Kähler manifold.

The geometrical meaning of ghosts of string theory is studied to construct a string field theory where ghost fields, $c(\sigma)$, and string coordinates, $x^{\mu}(\sigma)$, stand on a same footing. The BRST formalism for the open string field is rederived, while the Bowick and Rajeev proposal is modified. Namely, it is conjectured that the closed string field is a Kähler potential of the extended loop space, the set of maps, $S^1 \mapsto (x^{\mu}, \rho)$, where ρ is the conformal factor of a string worldsheet.

INTRODUCTION

It has been an ultimate goal of physicists to explain the diverse phenomena of the universe in a unified way. From Newton, Maxwell, Einstein, to Weinberg, Salam, and Glashow, we have discovered the underlying unified picture of nature. Now, string theory is believed to be a "theory of everything," since it incorporates almost all fashionable ideas ranging from axions and grand unification to supergravity and extra dimensions.[1,2] String theory is free of infinities and adjustable parameters. Above all it describes gravity in a quantum mechanically consistent way. But, unfortunately, string theory is consistent only in a critical dimension. For superstring, the critical dimension is 10. Since our universe has only four dimensions, it looks bizzare at first sight to have extra dimensions. But this misfortune turns out to be blessing. The extra dimensions give rich phenomena in a controllable manner, and allow us to calculate quantities like the number of generations of fermions and the ratio of some Yukawa couplings.[3] The topology of extra dimensions restricts strictly the phenomenology of string theory so that the low energy limit of string theory is almost uniquely determined.

Our understanding of string theory is far from completion. We do not understand enough to explain how to get four dimensions out of ten dimensions and why the cosmological constant of our universe is so small. Is superstring finite to all orders? Why are there three families in the standard model? These fundamental questions of string theory could be answered if we are able to formulate string field theory and to solve its field equations. Even though the perturbative approach in string theory *à la* Polyakov has brought into physics beautiful mathematics, *i.e. Riemann surface*,[4,5,6] we will not see nonperturbative effects in string theory by summing over all Riemann surfaces, unless we go beyond the perturbative regime. We may see the analog of this sympton in general relativity or in QCD. We may try to calculate the scattering amplitudes of massless spin two particle, namely graviton, to get the Schwarzshield solution in general relativity. But it is impossible to get a nontrivial background by accumulating spin two quanta or gravitational waves on the vacuum, because the nontrivial background is indeed a nonperturbative effect of the full Einstein equation. Similarly, in QCD we will not be able to see an interpolating field configuration called instanton between two different sectors of winding number by perturbing a vacuum of a fixed winding number. So it is very significant to construct string field theory to go beyond the perturbative regime and hopefully to shed light on fundamental questions of string theory. Especially the closed string field theory should be formulated in a background-geometry independent way[7,8,9], since it is a theory of gravity which tells us the geometry of spacetime.

So far our knowledge lacks a geometric principle to formulate string field theory, as the Riemannian geometry gives the geometrical setting for gravity. In this thesis an attempt is made to formulate a string field theory in a way which is free of background geometry, and the formulation heavily depends on the Kähler geometry of the loop space.

This dissertation is organized as follows. First, the salient feature of the geometric quantization[10] is described. The idea of geometric quantization is used to quantize the loop space, which is the configuration space of a closed string and also is the phase space of an open string modded out by zero modes.

The geometric quantization is also useful to get a unitary representation of non-compact group, *e.g.* $SL(2,R)$ or Virasoro group. Secondly, a field equation of closed string field is derived. That the loop space is an infinite dimensional Kähler manifold is re-proved, following the argument of Pressley.[11] The condition for an anomaly-free first quantized open string leads us to constuct geometrically a field equation for closed string field if we identify the Kähler potential of the loop space with the closed string field, which is natural because the loop space is the configuration space of closed string. Thirdly, this geometrical approach is extended to superstring, which is possible mainly because N=1 supermanifold can be equipped with a complex structure. Finally, the Faddeev-Popov ghosts of string theory enter on the same ground as the string coordinates, which are necessary to have a rich gauge symmetry in string field theory.[12] The usual BRST formalism[13,14,15,16,17] for open string field is rederived in this geometric approach. In appendix my work on Q-balls[18] is presented.

GEOMETRIC QUANTIZATION

Before explaining geometric quantization, let us study why a mathematical way of quantization is needed in physics. A classical system can be described either in a Hamiltonian formalism or in a Lagrangian formalism, which are interchangeable by a Legendre transformation. In a Hamiltonian formalism, the time evolution of the classical system is given by the Poisson bracket of Hamiltonian and observables of the classical system. The classical observables are functions on the phase space of the classical system. Geometrically speaking, a classical system is described by a symplectic manifold (M, ω), where the symplectic two form ω is closed and invertible and the bare manifold, M, corresponds to the phase space of the classical system. The bare manifold could have a nontrivial topology for a constrained system. The Poisson brackets of classical observables are defined as following:

$$\{f, g\}_{PB} = -\omega^{-1}(df, dg), \tag{2.1}$$

where f, g are functions on M and d is the exterior derivative. The Jacobi identity of Poisson brackets is guaranteed by the closedness of the symplectic two form, $d\omega = 0$. Since ω is invertible, it gives naturally a one-to-one correspondence between one forms and tangent vectors of M. For any tangent vector $u \in TM$, we have a unique one form, $a = \omega^{-1}(u) \in T^*M$, where TM, T^*M are the tangent space of M and its dual space, respectively. Especially, tangent vectors corresponding to the exact one forms are called *Hamiltonian Vector Fields*:

$$x_H = \omega^{-1}(dH). \tag{2.2}$$

The Lie drag of the symplectic two form along any Hamiltonian vector fields is zero due to the closedness of ω and the nilpotency of the exterior derivative, $d^2 = 0$:

$$\begin{aligned} \mathcal{L}_{x_H}\omega &= i_{x_H} d\omega + d i_{x_H}\omega \\ &= 0, \end{aligned} \tag{2.3}$$

where $i_{x_H}\omega$ is the interior product (inner product) of a form ω.

The operator i_v, for an arbitrary vector v, is defined as follow:

1) i_v is an antiderivation, which means that it satisfies the Leibnitz rule with Z_2 gradings to satisfy the antisymmetricity of a form ω.[19]
2) For any function (*i.e.* for a zero form) f, $i_v f = 0$
3) $i_v dx^i = v^i$, where v^i is the i-th component of v in a coordinate system.

A vector field v on a symplectic manifold (M,ω) is called locally Hamiltonian if its flow leaves ω invariant, that is, if

$$\mathcal{L}_v\omega = d i_v \omega = 0. \tag{2.4}$$

When $i_v\omega$ is not only closed but also an exact differential

$$i_v\omega = -dH, \tag{2.5}$$

v is said to be globally Hamiltonian; and H, determined up to an additive constant, is called the Hamiltonian. The number of independent global Hamiltonians determines the number of conserved quantites in a system. Energy conservation law is nothing but that the Hamiltonian H is preserved along an integral curve of the global Hamiltonian vector field, v: Let $f_t : M \mapsto M$ be a one-parameter map generated by v. Then

$$\frac{df_t}{dt} = v \text{ with } f_0(x) = x \in M \tag{2.6}$$

Since the Lie-drag of ω is zero, we find that $f_t^*\omega = \omega$ and

$$\frac{d}{dt}(H \circ f_t) = (dH)(v) = -\omega(v, v) = 0 \tag{2.7}$$

For a given classical system, we want to proceed to quantize the system. A standard way to quantize is following a canonical quantization. The quantum operator corresponding to a classical observable, $f(p, q)$, is constructed to be $f_{op}(-i\frac{\partial}{\partial q}, q)$ by replacing the conjugate variables by corresponding differential operators according to the canonical quantization. Normally this canonical quantization method works well, even though there is an ordering ambiguity in constructing the quantum operators. If the phase space has a nontrivial topology like the phase space of a constrained system, we need a coordinate-free way for quantization, because one coordinate patch does not cover whole phase space, and the canonical quantization rule applies only for a coordinate patch. Furthermore, there is no guarantee that we can construct a wave function on the phase space. Geometric quantization is an attemp to quantize without introducing a coordinate system. It was developed by Kostant and other mathematicians and generalized by Hess.[20] The latter method is called a metaplectic quantization.

Since quantum mechanics consists of states and quantum operators which act on the states, we will try to construct the states and the operators geometrically. The quantum states form a Hilbert space, which has a well-defined inner product, and the states are defined up to a phase; they form ray-representations for a underlying symmetry of the system, which are projective representations. Because of the phase ambiguity, the inner product is replaced by a Hermitean structure. Since in general the phase space does not allow a global coordinate patch, the Hilbert space can be thought of as a set of sections of a polarized

Hermitean line bundle over the phase space. The Hermitean line bundle should be polarized since otherwise the section would have too many degrees of freedom to satisfy the uncertainty principle. In order to quantize a system, we should at least be able to construct a Hermitean line bundle over the phase space of the system, which is endowed with a natural connection ∇. By natural we mean that it is compatible with the structure of the bundle, *i.e.* the Hermitean structure. Since the curvature of the bundle is required to be $\Omega = \frac{1}{2\pi}\omega$, we see that the phase space, M, is quantizable (*i.e.* a Hermitean line bundle exists over M) if the symplectic two form of the phase space is an integral element of the second cohomology of the phase space, M, $\omega \in H^2(M, Z)$, or equivalently, if the integration of ω over any closed, oriented 2-surface or more generally, over any integer 2-cycle in M is an integer. The cohomology class is an equivalence class among closed differential forms (=differential forms whose exterior derivatives are zero), where the equivalent relation is given as following; two closed forms μ and ν, $d\mu = 0$ and $d\nu = 0$, are in the same equivalent class if they differ by an exact form, $\mu = \nu + d\rho$. So the second cohomology class $H^2(M)$ can be written as a coset space, {closed two-forms on M}/{exact two-forms on M}. The integrality condition on the symplectic form can also be understood as a condition not to have global anomalies in the quantum theory. The integrality condition is not the only obstacle to quantization. For example, a disc is not quantizable, even though its topology is trivial (the symplectic form on the disc is exact, $d\omega = 0$). Since the disc is compact, the number of the quantum states of the system is finite, which is roughly the area of the disc divided by the Planck's constant due to the uncertainty principle. So it is impossible to quantize the disc in a way which varies smoothly with the area of the disc. In general, for a complex manifold,[21] it is possible to find a symplectic two form which satisfies the integrality condition.[22] If a manifold has such a symplectic two form, we can proceed with the geometric quantization in following steps.[23]

(1) Prequantization.

(2) Polarization for quantum theory.

Prequantization is to construct a Hermitean line bundle (called a prequantum bundle) with a connection of curvature $\frac{\omega}{2\pi}$ and prequantum operators acting on the sections of the bundle. The Hilbert space $\tilde{\mathcal{H}}$ in the prequantization construction is the space of all $s \in \Gamma_B(M)$, set of sections of the Hermitean line bundle for which the integral of $(s,s)\epsilon_\omega$ over M exists: The inner product on $\tilde{\mathcal{H}}$ is

$$\begin{aligned} \langle s_1, s_2 \rangle &= \frac{1}{(2\pi)^n} \int_M (s_1, s_2)\epsilon_\omega \\ &= \frac{1}{(2\pi)^n} \int_M s_1^\dagger s_2 \det \omega, \end{aligned} \tag{2.8}$$

where n is one half of the dimension of M and ϵ_ω is the Liouiville form of M. The operators acting on the prequantum Hilbert space $\tilde{\mathcal{H}}$ is constructed to be, for a classical observable f,

$$O_f = -i\nabla_{X_f} + f. \tag{2.9}$$

Then, because of the condition that the curvature of the bundle is $\Omega = \frac{1}{2\pi}\omega$, the prequantum operators satisfy the quantum analog of Poisson brackets,[24]

$$[O_f, O_g] = -iO_{\{f,g\}_{PB}} \tag{2.10}$$

as required by the correspondence principle. This completes the first step of the geometric quantization.

Clearly the above defined sections cannot describe quantum mechanics since they are sections of the whole phase space. In other words, in order to be in accord with the uncertainty principle we are required to eliminate half of the degrees of freedom. This elimination should be done globally. In geometric quantization this is achieved by assigning to each point of the phase space an n-dimensional subspace of the $2n$-dimensional tangent space of the phase space, M.

A polarization F is a subbundle of rank n, $F \subset TM^C$, complexified tangent space of M, which is

a) Integrable: if $u, v \in F$, then $[u, v] \in F$.

b) Maximally isotropic: for any $v_1, v_2 \in F_x$ where $x \in M$ $\omega_x^C(v_1, v_2) = 0$. (We donote F_x as the tangent vectors in F at a point $x \in M$. Similarly for ω.)

The condition a) guarantees by Frobenius theorem that the subspace F is locally integrable. The condition b) is needed to define a configuration space. With a polarization introduced above, we are able to construct a quantum Hilbert space, $\mathcal{H}$, as follows. $\mathcal{H}$ is a set of sections $\Psi \in \Gamma_B(M)$ which satisfy a polarization condition $\nabla_v \Psi = 0$ if $v \in F$. Since the connection preserves the inner product of the Hermitian line bundle, the quantum Hilbert space carries the same inner product as the prequantum Hilbert space but with a properly restricted integration domain. Obviously the prequantum operators are not well defined for the quantum Hilbert space, since they will kick out any element of the quantum Hilbert space if they act on the space. We have to project out the prequantum operators to get well-defined operators on the quantum Hilbert space. Now, let us define a projection operator P such that for any section $\Psi \in \Gamma_B(M)$, $P\Psi \in \mathcal{H}$ and $P^2 = 1$, *i.e.* $\nabla_v(P\Psi) = 0$ for any

$v \in F$. Then the quantum operator corresponding to a classical observable f is formally

$$r(f) = PO_f P^{-1}. \tag{2.11}$$

We see that for a given element in the quantum Hilbert space, $P\Psi \in \mathcal{H}$, $r(f)P\Psi$ belongs to the quantum Hilbert space, $\mathcal{H}$, since

$$\begin{aligned} r(f)P\Psi &= PO_f P^{-1} P\Psi \\ &= PO_f \Psi \in \mathcal{H}. \end{aligned}$$

A prequantum operator O_f preserves the polarization condition, if $[X_f, v] \in F$ for any $v \in F$: Suppose $\nabla_v \Psi = 0$ for any $v \in F$. Then $\nabla_v(O_f \Psi) = 0$, since

$$\begin{aligned} \nabla_v(O_f\Psi) &= \nabla_v(-i\nabla_{X_f} + f)\Psi \\ &= -i([\nabla_v, \nabla_{X_f}] - \nabla_{[v,X_f]})\Psi + (\nabla_v f)\Psi \\ &= -\omega(v, X_f)\Psi + (\mathcal{L}_v f)\Psi \\ &= 0, \end{aligned}$$

where in the last step the definition of the Hamitonian vector field is used. In this case the projection for the prequantum operator is trivial. But if the prequantum operator does not preserve the polarization condition, we have to find the explicit form of the projection operator.

STRING THEORY AS A KÄHLER GEOMETRY OF LOOP SPACE

Why Loop Space?

The spectrum of a closed string contains a massless spin-2 particle, which is not enough to say that the closed string has a graviton in it, unless we show that there is a gauge symmetry associated with the massless spin-2 particle. It has been shown that unitarity and Lorentz covariance implies that a massless particle of spin higher than 1/2 is not consistent, unless it is endowed with a gauge symmetry.[25] Since a closed string theory is consistent at a critical dimension, it better have a consistent spin-2 massless particle, graviton. It is also shown that in the zero slope limit one obtains from the Virasoro-Shapiro model the Einstein theory of graviton accompanied by a massless scalar field.[26,27,28]

Since the theory of gravity is described in terms of the Riemannian geometry of spacetime, one expects that the theory of closed string should also describe the geometry of some manifold. It is natural to expect that the underlying geometry will be a loop space. There are two reasons for us to believe that the loop space is important for the theory of closed string. The first reason can be given when we take an analogy with general relativity. General relativity is a theory of Riemannian manifold, which can be thought of as the set of all instantaneous states of a particle. In other words, General relativity describes the evolution of a Riemannian manifold, (M, g_{ab}), whose points correspond to point particles. Since the low energy limit of closed string is a point particle, the closed string will describe a manifold, whose points correspond to loops embedded into a Riemannian manifold.[29] The second reason is that a loop space is a configuration space of closed string and it forms the domain of closed string field. Since in a low energy limit the closed string describes gravity, the closed string theory is a theory of gravity. So we should not presuppose the geometry of spacetime. The Sobolev spaces $\Omega_s R^n$ for s a positive integer are defined to be the set of all maps x^μ such that first s derivative of x^μ are square integrable:

$$\Omega_s R^n = \{x^\mu(\sigma) : \int_0^{2\pi} \frac{d^s}{d\sigma^s} x^\mu(\sigma) \frac{d^s}{d\sigma^s} x_\mu(\sigma) d\sigma < \infty, \sigma \in S^1\} \tag{3.1}$$

And also, this Sobolev space $\Omega_s R^n$ can be generalized for any real number s. It is known that for any choice of Sobolev space the loop space is independent of the dimension of spacetime and furthermore for an appropriate class of loops, the loop space is essentially independent of all the topology of the spacetime.[30]

The Idea of Bowick and Rajeev's Proposal

The loop space of a Minkowski space is known as an infinite dimensional Kähler manifold.[11,31] Since the Kähler potential of the loop space is certainly a functional of loops and is Hermitean, Bowick and Rajeev[7] conjectured that the closed string field is the Kähler potential of the loop sapce. They proposed that an equation for the closed string field be the equation which guarantees for an open string to be consistent in a quantum mechanical sense, by noting that the loop space is a phase sapce of open string modded out by zero modes if we use a change of variable:

$$x'^\mu(\sigma) = \begin{cases} (p + q')(\sigma), & \text{for } 0 \le \sigma \le \pi, \\ (p - q')(-\sigma), & \text{for } -\pi \le \sigma \le 0, \end{cases} \tag{3.2}$$

where $q(\sigma)$ is an open string variable and $p(\sigma)$ is its conjugate variable and $'$ denotes the derivative with respect to σ.

In order for the quantum theory of open string to be consistent, the quantization procedure should preserve the classical symmetry of open strings.[32] Since the classical theory of open strings has reparametrization invariance, *i.e.* diffeomorphism of S^1 or $Diff(S^1)$, to see whether the quantum theory is consistent we may consider a bundle over $Diff(S^1)/S^1$. [33] We modded out $Diff(S^1)$ by S^1, because S^1, the global rotation of a circle, is trivially preserved in the theory when we quantize an open string. If we take the fiber as a quantum Hilbert space of open string, the curvature of the bundle should be zero in order to have a covariantly constant section, in other words to have a reparametrization-invariant state in the quantum Hilbert space of open string, which is going to be identified as a vacuum state. The bundle turns out to be a holomorphic line bundle, since the physical state condition for an open string can be thought of as a holomorphicity. But as it is well known, in a covariant formalism the open string by itself is not consistent, unless the Faddeev-Popov ghosts are incorporated.[34] In addition to the holomorphic line bundle, Bowick and Rajeev considered the canonical line bundle over $Diff(S^1)/S^1$, which is the wedge product of all one form bases of $Diff(S^1)/S^1$, $\wedge^{max} T^* Diff(S^1)/S^1$, interpreting the canonical line bundle as a reparametrization ghost vacuum. With this setting, the proposed equation for the closed string field is that the curvature of the twisted bundle between the holomorphic bundle and the canonical bundle over $Diff(S^1)/S^1$, is zero, since it gives an equation for the Kähler potential of the loop space. Also they were able to show that the d=26 Minkowski space is indeed a solution for the equation and to derive the source-free Einstein equation, *i.e.* $R_{\mu\nu} = 0$ by expanding in a normal coordinate a background metric $g_{\mu\nu}$. Several solutions for the equation are found.[35,36,37]

The Loop Space of A Riemannian Manifold Is A Kähler Manifold

A Kähler manifold is a complex and symplectic manifold. A manifold is complex, if it is endowed with a complex structure, J, which is a tensor and whose square is -1:

$$J_a^b J_b^c = -\delta_a^c. \tag{3.3}$$

A symplectic manifold is endowed with a symplectic structure, ω, which is a nondegenerate closed two form and invertible:

$$d\omega = 0. \tag{3.4}$$

If a manifold is endowed with both a complex structure and a symplectic structure, which are compatible with each other, called Kähler structure, it is a Kähler manifold. The compatibility condition is as follows: Suppose M is a Kähler manifold with a Kähler structure (ω, J). Then for all $x \in M$,

(a) $\omega_x(J_x u, J_x v) = \omega_x(u, v)$ for all $u, v \in T_x M$

(b) $g_x(u, v) = \omega_x(u, J_x v)$ is a positive definite inner product on $T_x M$.

The Kähler form on T_x is given by

$$(u, v) \mapsto g_x(u, v) + i\omega_x(u, v). \tag{3.5}$$

Now let us try to assign a complex structure on the based loop sapce, $\Omega M \simeq LM/M$, which is a set of all loops with a common point embedded into M: We modded out the loop space, LM, by the set of constant maps, *i.e.* M itself, since the constant maps are not interesting because of the translational invariance. In a local coordinate chart, the complex structure gives a natural splitting of the coordinates into two categories. For example, if we assign a complex structure on R^2 to get C^1, then in a local coordinate chart we have a natural splitting, $J : (x, y) \in R^2 \mapsto (z, \bar{z}) \in C^1$ where $z = x + iy$ and

$\bar{z} = x - iy$. Take a point in a coordinate chart, $U_\alpha \subset \Omega M$, for the based loop space, $x^\mu(\sigma) \in U_\alpha$, and expand it in a Fourier series

$$x^\mu(\sigma) = \sum_{n \neq 0} x_n^\mu(\sigma) e^{in\sigma},$$

where the complex numbers x_n^μ satisfy the reality condition, $x_{-n}^\mu = \bar{x}_n^\mu$, for $x^\mu(\sigma)$. Then we can define a map for the coordinate chart U_α such that for any $x^\mu(\sigma) \in U_\alpha$

$$J : x^\mu(\sigma) \mapsto (Jx^\mu)(\sigma) = \sum_{n>0} i x_n^\mu e^{in\sigma} - \sum_{n<0} i x_n^\mu e^{in\sigma} \tag{3.6}$$

Note that J is a well-defined map on U_α, since $Jx^\mu(\sigma)$ belongs to U_α($(Jx^\mu)^*(\sigma) = (Jx^\mu)(\sigma)$).

The coordinate system for a based loop space, Ω, could be $\{x^\mu(\sigma)\}$ or equivalently $\{x_n^\mu\}$. The transformation function between two coordinate systems are $e^{in\sigma}$: $x^\mu(\sigma) = \sum_{n\neq 0} x_n^\mu e^{in\sigma}$. The corresponding bases for the tangent vectors are $\frac{\partial}{\partial x^\mu(\sigma)}$ or $\frac{\partial}{\partial x_n^\mu}$. The transformation functions are again $e^{in\sigma}$. But to study a global property of Ω we need a global basis. One choice is $e^{in\sigma}\frac{d^\alpha}{d\sigma} = e^{in\sigma}(\frac{d}{d\sigma}, \cdots, \frac{d}{d\sigma}), \alpha = 1, \ldots, d = \dim M$. This choice is natural if we note that the loop space $\{S^1 \mapsto R \times \cdots \times R\}$ can be thought of as a Lie-algebra of loop group $\{S^1 \mapsto U(1) \times \cdots \times U(1)\}$, since the Lie-algebra of $U(1)$ is R. ($e^{in\sigma}\frac{d^\alpha}{d\sigma}$ is an element of the Lie-algebra of the loop group.) If we take $e^{in\sigma}\frac{d^\alpha}{d\sigma}$ for $n > 0$ as a holomorphic basis for $T^C\Omega$, the complexified tangent space of the based loop space, Ω, and $e^{-in\sigma}\frac{d^\alpha}{d\sigma}$ for $n < 0$ as an antiholomorphic basis, then the complex structure is integrable, since $[e^{in\sigma}\frac{d^\alpha}{d\sigma}, e^{im\sigma}\frac{d^\beta}{d\sigma}]$ for $n, m > 0$ is again holomorphic:

$$[e^{in\sigma}\frac{d^\mu}{d\sigma}, e^{im\sigma}\frac{d^\nu}{d\sigma}] = \eta^{\mu\nu} i(m-n) e^{i(m+n)\sigma}\frac{d^\mu}{d\sigma} \text{ (no sum over } \mu) \tag{3.7}$$

Our next step is to define a left invariant symplectic form ω on Ω. According to Pressley[11], it is given by, for $u, v \in T^C\Omega$,

$$\omega(u,v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma u^\mu(\sigma)\frac{D}{d\sigma}v^\nu(\sigma)g_{\mu\nu} \tag{3.8}$$

or equivalently

$$\omega(u,v) = \sum_{n\neq 0} inu_n^\mu \bar{v}_n^\nu g_{\mu\nu} \tag{3.9}$$

where $g_{\mu\nu}$ is the metric tensor of the target space and D is the general covariant derivative, $Dg_{\mu\nu} = 0$. We denote as $\frac{D}{d\sigma} \equiv \frac{dx^\mu(\sigma)}{d\sigma}D_\mu$ the covariant derivative along the tangent to the loop. We may write a tangent vector in different bases as follows:

$$\begin{aligned} u &= u^\mu(\sigma)\frac{\partial}{\partial x^\mu(\sigma)} \\ &= \tilde{u}_n^\mu \frac{\partial}{\partial x_n^\mu} \\ &= u_n^\mu e^{in\sigma}\frac{d^\mu}{d\sigma}. \end{aligned} \tag{3.10}$$

Clearly ω is nondegenerate, since the constant maps are modded out. And it is closed due to the Bianchi identity for the Riemann curvature tensor of the target space. The exterior derivative of the symplectic two form is for any tangent vectors, $u, v, t \in T^C\Omega$,

$$d\omega(u,v,t) = \mathcal{L}_u\omega(v,t) - \mathcal{L}_v\omega(u,t) + \mathcal{L}_t\omega(u,v) - \omega([u,v],t) + \omega([u,t],v) - \omega([v,t],u). \tag{3.11}$$

The first three terms are zero since ω is left invariant. And the remaining three terms are zero due to Jacobi identity. So far we have constructed a complex structure J and a symplectic structure ω on Ω, the based loop space of R^n. But to prove that Ω is a Kähler manifold, we have to show that two structures are compatible with each other. From the explicit form of ω in equation (3.9), we easily see that the compatibility conditions are satisfied for the based loop space. Note that

$$Ju_n = i\mathrm{sgn}(n)u_n,$$

where $\mathrm{sgn}(n) = 1\ (-1)$ if $n > 0\ (n < 0)$. By the direct calculation, we find

$$\begin{aligned}
\omega(Ju, Jv) &= \sum_{n\neq 0}(i\mathrm{sgn}(n))u_n^\mu(i\mathrm{sgn}(-n))\bar{v}_n^\nu g_{\mu\nu} \\
&= \sum_{n\neq 0} u_n^\mu \bar{v}_n^\nu g_{\mu\nu} \\
&= \omega(u, v),
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\omega(u, Ju) &= \sum_{n\neq 0} inu_n^\mu(i\mathrm{sgn}(-n))\bar{u}_n^\nu g_{\mu\nu} \\
&= \sum_{n>0} nu_n^\mu \bar{u}_n^\nu g_{\mu\nu} - \sum_{n<0} nu_n^\mu \bar{u}_n^\nu g_{\mu\nu} \\
&= \sum_{n>0} n(u_n^\mu \bar{u}_n^\nu + \bar{u}_n^\mu u_n^\nu) g_{\mu\nu} \\
&> 0,
\end{aligned} \tag{3.13}$$

provided that the metric $g_{\mu\nu}$ has an Euclidean signature.

By constructing a Kähler structure on the loop space of R^n, we indeed see that the loop space of Riemannian manifold is a Kähler manifold, since locally we can think of the Riemannian manifold as R^n with additional structure (=metric), which is certainly compatible with the complex structure we constructed. The origin of the symplectic form ω on the based loop group Ω is explained by Segal.[38] He noted that the symplectic two form originated from the symplectic structure of the coadjoint orbit of the virasoro group.

Consider a $U(1)$ central extension $\hat{M}$ of a loop group M, the space of all maps $f : S^1 \mapsto G$ lying in the Sobolev class H^1.

$$1 \to S^1 \to \hat{M} \to M \to 1 \tag{3.14}$$

Let a be an element of the coadjoint representation of the Lie algebra of $\hat{M}$, $L(\hat{M})$, which lies inside $1 \oplus L(M) \subset R \oplus L(M) = L(\hat{M})$. Then for any $f \in M$, a transforms as follows under the affine action of M on $L(M)$:

$$f : a \mapsto (Ad_f)a + f^{-1}f' \tag{3.15}$$

The stabilizer group, T, consists of elements such that $f' = 0$ and $Ad_f = 1$. So f is constant or in fact $f = 1$, since we are considering an element in the based loop space ($f(0) = 1$). Therefore the stabilizer group T is G itself (=constant maps in M). Hence $M/G \simeq \Omega$ is a coadjoint orbit of $L(\hat{M})$. Then by Kirillov's construction the coadjoint orbit has a natural symplectic structure.[39]

Now let us construct the symplectic structure on a coadjoint orbit for a Lie group G. Let $\mathcal{G}$ be the Lie algebra of a Lie group G. An adjoint representation shall be denoted as $\mathcal{G}_{Adj} \equiv (\mathcal{G})$. The element in the adjoint representation transforms under an action of a group element $g \in G$ as follows: for any $u \in (\mathcal{G})$

$$g(u) = [g, u], \tag{3.16}$$

where g in the right hand side is the element in the adjoint representation corresponding to $g \in G$. Now let $(\mathcal{G})_{Coadj} \equiv (\mathcal{G})^*$ be the coadjoint representation. The action of $g \in G$ on $a \in (\mathcal{G})^*$ is determined by the requirement that $a \in (\mathcal{G})^*$ will map $u \in (\mathcal{G})$ into a group singlet (=constant), $a(u) \in C$:

$$\begin{aligned}(g(a))(u) &\equiv -a(g(u)) \\ &= -a([g, u]).\end{aligned} \tag{3.17}$$

For a finte group the coadjoint representation is isomorphic to the adjoint representation because the finite group has a well-defined bilinear form given by the Cartan-Killing metric. But for an infinite Lie group the Cartan-Killing metric is not well defined. In other words, for the infinite Lie group there is no isomorphism between the adjoint representation and coadjoint representation. For the Virasoro group, a quadratic differential is an element of the coadjoint representation while a weight -1 vector is an element of the adjoint representation. Let b be a fixed coadjoint vector ($\equiv$ an element of the coadjoint representation of $\mathcal{G}$) and W_b be the orbit of b under the action of G. W_b is a coadjoint orbit. Now we can define G-invariant symplectic structure on W_b: for any $a, a\prime \in TW_b$ there must be adjoint vectors which send b into $a, a\prime$, $u(b) = a, u\prime(b) = a\prime$, where $u, u\prime$ are the group elements. If we define a symplectic structure as

$$\omega(a, a\prime) = b([u, u\prime]), \tag{3.18}$$

then it is elementary to show that ω defined is antisymetric, G-invariant, nondegenerate, and exact. So far we have seen that the based loop space is a Kähler manifold. Using the Kähler structure we proceed to quantize an openstring in a geometrical way.

Bowick and Rajeev's idea to get a background-independent equation for the closed string field is as following; first conjecture that the closed string field be the Kähler potential for a loop space of an arbitrary Riemannian manifold, then find an equation for the Kähler potential for a given background by requiring the consistent quantum theory of open string for the given background geometry and try to get by eliminating the background dependence a general setting which does not assume any background dependence.

They proposed that *the closed string field is the Kähler potential of the loop space of a Riemannian manifold and the equation for the closed string field be the curvature of the twisted line bundle of the holomorphic line bundle and the canonical line bundle over* $Diff(S^1)/S^1$.

It will be possible to get the true ground state of the closed string field by comparing the ground states, if we are able to construct the action for the closed string field, which leads to the equation proposed above. To find the action it will be important to study the property of the moduli space of the connections of the holomorphic line bundle over $\mathcal{D}iff(S^1)/S^1$, which is the space of all connections of the holomorphic line bundle over $\mathcal{D}iff(S^1)/S^1$ modded out by $U(1)$, the structure group of the bundle.[40]

SUPERSTRING

Geometric Quantization of The Superstring

We wish to describe the closed superstring in terms of the Kähler geometry of superloop space, the space of maps from the circle S^1 to the supermanifold $\tilde{R}^{d-1,1}$, with periodic boundary conditions in the bosonic or antiperiodic boundary conditions in the fermionic coordinates. In order to apply the geometric quantization techniques to this system it is convenient to think of superloop space as the phase space of the open superstring modded out by zero modes. To define the configuration space we consider the space of maps

$$\tilde{q} : [0, \pi] \mapsto \tilde{R}^{d-1,1}, \tag{4.1}$$

where $\tilde{R}^{d-1,1}$ is a supermanifold. These maps may be written as $\tilde{q} = (q_B, q_F)$ where q_B and q_F refer to bosonic and fermionic coordinates. As in the bosonic case q_B satisfies $\partial_\sigma q_B = 0$ for $\sigma = 0, \pi$ and q_F is wrtten as a spinor distribution with two spinor components q_+ and q_-.

The boundary conditions on q_F are

$$q_+(0) = q_-(0), \tag{4.2}$$

$$q_+(\pi) = \epsilon q_-(\pi), \tag{4.3}$$

The parameter ϵ specifies the Neveu-Schwarz ($\epsilon = -1$) and Ramond ($\epsilon = +1$) sectors of the theory.

As in the case of bosonic strings it is convenient to redefine the interval to $[-\pi, \pi]$ and map

$$\tilde{q} : [-\pi, \pi] \mapsto \tilde{R}^{d-1,1}.$$

In this way we define loops in $\tilde{R}^{d-1,1}$.

The coordinates in the classical phase space are $(q_B, q_F; P_B, P_F)$, where P_B and P_F are the bosonic and fermionic conjugate momenta respectively. On the above extended interval we may define (suppressing the spacetime indices)

$$x'(\sigma) = \begin{cases} (P_B + q'_B)(\sigma), & 0 \leq \sigma \leq \pi, \\ (P_B - q'_B)(-\sigma), & -\pi \leq \sigma \leq 0, \end{cases} \tag{4.4}$$

$$\psi(\sigma) = \begin{cases} (\epsilon S P_F + q_F)(\sigma), & 0 \leq \sigma \leq \pi, \\ (P_F + \epsilon S q_F)(-\sigma), & -\pi \leq \sigma \leq 0. \end{cases} \tag{4.5}$$

The $'$ denotes a derivative with respect to σ and the matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is needed to take care of the boundary conditions for the fermionic strings. x' and ψ form components of $\tilde{q}$ which maps S^1 to loops in a supermanifold. Hence superloop space is the phase space of open superstrings up to zero modes of bosonic strings. To proceed with the geometric quantization we need to display the symplectic form ω. This ω is given by

$$\omega(u, v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma \int d\theta u^\mu D v^\nu \eta_{\mu\nu}, \tag{4.6}$$

where $u = u_B + \theta u_F$ is a tangent vector of superloop space and θ is an anticommuting variable. The sympectic structure is manifestly invariant under supersymmetry, since we are using a supersymmetric covariant derivative $D = -i\frac{d}{d\theta} + \theta\frac{d}{d\sigma}$. It is strightforward to show that $d\omega = 0$ and ω is nondegenerate.

We can expand the vectors u and v in their normal modes by writing

$$u^\mu(\sigma) = \sum_{n=-\infty}^{\infty} u_n^\mu e^{in\sigma}, \tag{4.7}$$

$$v^\mu(\sigma) = \sum_{n=-\infty}^{\infty} v_n^\mu e^{in\sigma}. \tag{4.8}$$

Then

$$\omega(u,v) = \sum_{n=-\infty}^{\infty} (-i)(-)^{g(u^B)} n (u_B)^{\mu}_{n} (v_B)^{\nu}_{-n} \eta_{\mu\nu} + \sum_{n=-\infty}^{\infty} (-i)(u_F)^{\mu}_{n} (v_F)^{\nu}_{-n} \eta_{\mu\nu}. \tag{4.9}$$

The Z_2 grading $g(u_B)$ which is defined to be zero for bosonic variable and 1 for fermionic variable appears because the bosonic part of a tangent vector u can be either a commuting or anticommuting variable.

Once the symplectic stucture ω is given the Poisson bracket of any two functions f and g can be evaluated as $\{f,g\} = -\omega^{-1}(df, dg)$. Since the phase space of open superstring can be thought of as a superloop space (up to zero modes), the diffeomorphism of superloop (or super- Virasoro group) is the symmetry of open superstring. This symmetry is realized through the Poisson algebra of the phase space of open superstring. One can find functions λ_n and ϕ_n that provide a representation for the super-Virasoro algebra

$$\begin{aligned} \{\lambda_m, \lambda_n\} &= -i(m-n)\lambda_{m+n}, \\ \{\lambda_m, \phi_n\} &= -i(\frac{1}{2}m - n)\phi_{m+n}, \\ \{\phi_m, \phi_n\} &= -2i\lambda_{m+n}, \end{aligned} \tag{4.10}$$

where $m, n \in Z$. We shall work out explicitly the Ramond sector (R sector) of the theory and just quote the results for the Neveu-Schwarz sector (NS sector). In the NS sector the indices of the Grassman valued quantities (like ϕ_n) run over the set of half integers. The functions satisfying (4.10) are given by

$$\lambda_n = -\frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}[(x')^2 + i\psi\psi'], \tag{4.11}$$

$$\phi_n = \frac{i}{2\pi}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}\psi x'. \tag{4.12}$$

Using the mode expansion of x and ψ

$$x(\sigma) = \sum_{n=-\infty}^{\infty} x_n(e^{in\sigma} - e^{in\pi}), \tag{4.13}$$

$$\psi(\sigma) = \sum_{n=-\infty}^{\infty} \psi_n(e^{in\sigma} - e^{in\pi}), \tag{4.14}$$

we can write a mode decomposition of λ_n and ϕ_n as

$$\lambda_n = -\frac{1}{2}\sum_{p=-\infty}^{\infty} p(n+p)x_p x_{-n-p} - \frac{1}{2}\sum_{p=-\infty}^{\infty} p\psi_p\psi_{-p-n}, \tag{4.15}$$

$$\phi_n = \sum_{p=-\infty}^{\infty} (p+n)\psi_p x_{-n-p}. \tag{4.16}$$

The next step is the construction of the Hamiltonian vector fields X_{λ_n} and X_{ϕ_n} and the corresponding operators O_{λ_n} and O_{ϕ_n} which act on the prequantum Hilbert space. Afterwards, we need to introduce a polarization. Since we are dealing with a Kähler manifold, we will use its complex structure to define it. The complex structure J on the complexified tangent space of the based superloop space can be written as

$$J(X) = -i\sum_{n\neq 0} \mathrm{sgn}(n)[X_n^{B\mu}\frac{\partial}{\partial x_n^\mu} + X_n^{F\mu}\frac{\partial}{\partial \psi_n^\mu}], \tag{4.17}$$

where $\mathrm{sgn}(n) = 1\ (-1)$ if $n > 0\ (n < 0)$. Then $J(X) = \mp iX$ if X is of type $(1,0)$ and $(0,1)$ respectively (*i.e.* $n < 0$ or $n > 0$ respectively.) By construction $J^2 = -1$. Given the complex structure the natural (Kähler) choice for the polarization is

$$\mathcal{L}_v\Psi = 0 \text{ for } v \in \mathcal{P} = \{(1,0) - \text{type vectors},\} \tag{4.18}$$

where Ψ is a (square integrable) section of the Hermitian line bundle over the based superloop space. In a coordinate system as the one used in (4.17) we can regard the polarization as spanned by all the vectors $\{\partial/\partial x_n, \partial/\partial\psi_m$ with $n, m >$

$0\}$. The wave function, Ψ can then be written as $\Psi = e^{-K/2}\tilde{\Psi}$ with $\tilde{\Psi}$ a function of $\bar{x}_n$ and $\bar{\psi}_m$ only. The Hamiltonian vector fields of functions λ_n, ϕ_n preserve the polarization only if $n \geq 0$. In other words, $\mathcal{L}_v X_{\lambda_n}$ and $\mathcal{L}_v X_{\phi_n}$ belong to $\mathcal{P}$, the polarization only if $n \geq 0$.

For all functions f which preserve the above polarization we may construct a quantum operator acting on the quantum Hilbert space $r(f) = O_f$ *i.e.*

$$r(f)\Psi = -i\nabla_{X_f}\Psi + f\Psi. \tag{4.19}$$

It is straightforward to show that $r(f)\Psi$ satisfies the polarization if Ψ does. Upon computation we find, choosing a gauge for the connection A on the Hermitian bundle such that A_p is zero for $p > 0$,

$$\begin{aligned} r(\lambda) = &-\sum_{p>0}(p+n)\bar{x}_{p+n}\frac{\partial}{\partial\bar{x}_p} - \frac{i}{2}\sum_{p>0}(2p+n)\bar{\psi}_{p+n}\frac{\partial}{\partial\bar{\psi}_p} \\ &+\frac{1}{2}\sum_{p=1}^{n}p(n-p)\bar{x}_p\bar{x}_{n-p} + \frac{1}{2}\sum_{p=1}^{n}(\frac{1}{2}n-p)\bar{\psi}_{n-p}\bar{\psi}_p \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} r(\phi_n) = &-\sum_{p>0}\bar{\psi}_{n+p}\frac{\partial}{\partial\bar{x}_p} + \sum p > 0(n+p)\bar{x}_{n+p}\frac{\partial}{\partial\bar{\psi}_p} \\ &+\frac{1}{2}\sum_{p=1}^{n}p\bar{\psi}_{n-p}\bar{x}_p + \frac{1}{2}\sum_{p=1}^{n}(n-p)\bar{\psi}_p\bar{x}_{n-p}. \end{aligned} \tag{4.21}$$

Also for $n = 0$, we get

$$r(\lambda_0) = -\sum_{n>0}n\bar{x}_n\frac{\partial}{\partial\bar{x}_n} - \sum_{p>0}p\bar{\psi}_p\frac{\partial}{\partial\bar{\psi}_p} + ia, \tag{4.22}$$

$$r(\phi_0) = -\sum_{p>0}\bar{\psi}_p\frac{\partial}{\partial\bar{x}_p} + \sum_{p>0}p\bar{x}_p\frac{\partial}{\partial\bar{\psi}_p}. \tag{4.23}$$

The constant term $(+ia)$ was added in eq.(4.22) to amount for possible normal ordering ambiguities. a is the ground state energy. Since the Hamiltonian vector fields associated with the negative n's are not holomorphic, the prequantum operators corresponding to classical observables, λ_n, ϕ_n for $n < 0$ do not preserve the polarization condition. So we need an indirect method to quantize those observables. By using a method similar to Bowick and Rajeev, we can find the corresponding quantum operators for these observables. First note that since the polarization changes if we change the complex structure, so do the operators $r(\lambda_n)$ and $r(\phi_n)$. A change of the complex structure can be seen as a motion on Super $\mathcal{D}iff(S^1)/S^1$, since reparametrizations of the superloops, except for the rigid rotation (for NS sector) or for the rigid super S^1 (for R sector), do not leave the complex structure invariant. The alternative Hilbert space for the superstring thus forms a holomorphic line bundle over Super $\mathcal{D}iff(S^1)/S^1$. Let L_n, F_n be tangent vectors at the origin of Super $\mathcal{D}iff(S^1)/S^1$ that generate the superconformal algebra. If we require that $r(\lambda_n)$ for a different complex structure be the left translation of $r(\lambda_n)$ at the origin, we can define a metric connection by $\nabla_{L_n} = \mathcal{L}_{L_n} + r(\lambda_n)$, and analogously $\nabla_{F_n} = \mathcal{L}_{F_n} + r(\phi_n)$. Now, since the connection is compatible with the metric (or the inner product) of the bundle, we have for any sections ψ, ϕ,

$$\nabla_{L_n}\langle\psi,\phi\rangle = \langle\nabla_{L_n}\psi,\phi\rangle + \langle\psi,\nabla_{L_n}\phi\rangle.$$

But since the covariant derivative for an inner product is same as the Lie derivative for the inner product and any tangent vectors are Killing vectors for Super $\mathcal{D}iff(S^1)/S^1$, which is a homogeneous space, we also have

$$\nabla\langle\psi,\phi\rangle = \mathcal{L}_{L_n}\langle\psi,\phi\rangle = \langle\mathcal{L}_{L_n}\psi,\phi\rangle + \langle\psi,\mathcal{L}_{L_n}\phi\rangle.$$

So we find

$$r(\lambda_{-n}) = -r(\lambda_n)^{\dagger}.$$

And similarly we get

$$r(\phi_{-n}) = +r(\phi_n)^{\dagger}.$$

Therefore, we have quantum operators for λ_n and ϕ_n with $n < 0$. Explicitly,

$$\begin{aligned} r(\lambda_{-n}) =& \sum_{p>0} p\bar{x}_p \frac{\partial}{\partial \bar{x}_{p+n}} + \frac{1}{2}\sum_{p>0}(2p+n)\bar{\psi}_p \frac{\partial}{\bar{\psi}_{p+n}} \\ & - \frac{1}{2}\sum_{p=1}^{n} \frac{\partial}{\partial \bar{x}_{n-p}} \frac{\partial}{\bar{x}_p} - \frac{1}{2}\sum_{p=1}^{n}(\frac{n}{2} - p)\frac{\partial}{\partial \bar{\psi}_p}\frac{\partial}{\partial \bar{\psi}_{n-p}}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} r(\phi_{-n}) =& -\sum_{p>0} p\bar{x}_p \frac{\partial}{\bar{\psi}_{n+p}} + \sum_{p>0} \bar{\psi}_p \frac{\partial}{\partial \bar{x}_{n+p}} \\ & + \frac{1}{2}\sum_{p=1}^{n} \frac{\partial}{\partial \bar{\psi}_{n-p}} \frac{\partial}{\partial \bar{x}_p} + \frac{1}{2}\sum_{p=1}^{n} \frac{\partial}{\partial \bar{\psi}_p}\frac{\partial}{\partial \bar{x}_{n-p}}. \end{aligned} \tag{4.25}$$

The action of Super $\mathcal{D}iff(S^1)/S^1$ in the superloop space is tantamount to a change of the complex structure used to polarize the phase space, and thus to a change in the quantum operators $r(\lambda_n), r(\phi_n)$. As mentioned before, such a change defines a connection through $\nabla_{L_n} = \mathcal{L}_{L_n} + r(\lambda_n), \nabla_{F_n} = \mathcal{L}_{F_n} + r(\phi_n)$. If the connection has curvature, the quantum theory has an anomaly. The curvature is given, in the R sector, by

$$\begin{aligned} F(L_n, L_m) &= [\nabla_{L_n}, \nabla_{L_m}] - \nabla_{[L_n, L_m]} \\ &= [r(\lambda_n), r(\lambda_m)] - r(\{\lambda_n, \lambda_m\}) \\ &= (\frac{1}{8}dn^3 - 2an)\delta_{n,-m}, \end{aligned} \tag{4.26}$$

$$\begin{aligned} F(F_n, F_m) &= [r(\phi_n), r(\phi_m)]_+ - r(\{\phi_n, \phi_m\}) \\ &= (\frac{1}{2}dn^2 - 2a)\delta_{n,-m}, \end{aligned} \tag{4.27}$$

where d is the number of spacetime dimensions. Recall that a is an arbitrary additive constant term in $r(\lambda_0)$.

In the NS sector the result is

$$F(L,L_m) = \left[\frac{1}{8}d(n^3 - n) - 2an\right]\delta_{n,-m}, \tag{4.28}$$

$$F(F_r, F_s) = \left[\frac{1}{2}d(r^2 - \frac{1}{4}) - 2a\right]\delta_{r,-s}, \tag{4.29}$$

where r, s are half integers.

From eqs.(4.26),(4.27),(4.28) and (4.29) we see that the Hermitian operators do not form a representation to super-$\mathcal{D}iff(S^1)$. The r.h.s. indeed represents the anomaly in the super-Virasoro algebra. However, since this anomaly is 2-cocycle, these operators form a projective representation of super-$\mathcal{D}iff S^1$. Throughout, the Kähler potential K stands on special ground since it induces the sympectic structure. It is for this reason that K is identified with the closed string or superstring field. As it stands, however, we still do not have a gauge invariant description due to the anomaly. We must therefore find a geometric description of the ghost sector. In the next section we compute the curvature of the canonical line bundle over Super $\mathcal{D}iff(S^1)/S^1$. This is interpreted as the ghost contribution to the representation. We then put two bundles together to form the flat background solution to the equation for the closed superstring.

Curvature of the Canonical Line Bundle

The physical processes of the string theory will of course be independent of the way the string is parametrized. As we have just seen, canonical transformations can induce a different parametrization and change the complex structure. Thus, a particular complex structure carries with it a parametrization. By applying super$\mathcal{D}iff(S^1)$ to J we can effectively generate all possible parametrizations. However, super-$\mathcal{D}iff(S^1)$ contains generators which leave the complex structure invariant. Indeed, for the R sector the action of L_0 and

F_0 on J leaves it invariant, while for the NS sector the action of L_0 only leaves J invariant. In either case we will call Super $\mathcal{D}iff(S^1)/S^1$ the set of all complex structure of the superloop space. We will consider Super $\mathcal{D}iff(S^1)/S^1$ as an abstract manifold. By studying the properties of its curvature we can extract informations on the anomaly structure of open superstring. Then combining the holomorphic Hermitian line bundle with the canonical line bundle we can describe a gauge invariant quantum theory of open superstring, which also gives an equation for the closed superstring.

The manifold $\mathcal{M} \equiv$Super $\mathcal{D}iff(S^1)/S^1$ is a homogeneous and Kähler manifold since it is a coset space and has a Kähler structure. The tangent vectors of $\mathcal{M}$ form the $(N = 1)$ superconformal algebra,

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m}, \\
[L_n, F_m] &= (\frac{1}{2}n - m)F_{n+m}, \\
[F_n, F_m]_+ &= 2L_{n+m},
\end{aligned} \tag{4.30}$$

and carry the complex structure J given by

$$JG_m = \begin{cases} iG_m \text{ if } m < 0 \\ -iG_m \text{ if } m > 0, \end{cases}$$

where $m \in Z$, $G_m \in \{L_m, F_m\}$ for the R sector, and $G_m \in \{L_m, F_{m+\frac{1}{2}}\}$ for the NS sector. As in the previous section, we write all intermediate for the R sector and quote the results for both sectors.

The most general solution to the closure condition for the Kähler 2-form is given by[41,38]

$$\begin{aligned}
\omega(L_m, L_n) &= (am^3 + bm)\delta_{m,-n}, \\
\omega(L_m, F_n) &= 0, \\
\omega(F_m, F_n) &= (4am^2 + b)\delta_{m,-n},
\end{aligned} \tag{4.31}$$

where a, b are arbitrary constant. Note that ω is not invertible for $-b/a = m^2$ for nonzero intergers m.

Let us consider $\mathcal{M}$ as a graded flag manifold G/H and exploit its complexification by splitting it into M_+ and M_-. Then we can take the decomposition of the associated Lie algebra of G as

$$\underline{G}_c = \underline{H}_c + \underline{M}_+ + \underline{M}_-$$

where

$$\underline{H}_c = \{L_0, F_0\}$$

.

Since $\mathcal{M}$ is a flag manifold, we can calculate the curvature of $\mathcal{M}$ by using the Toeplitz operator method.[42]

For any tangent vector, X we may define the Toeplitz operator

$$\phi(X) = \nabla_X - \mathcal{L}_X,$$

where ∇_X is a covariant derivative and $\mathcal{L}_X$ is the Lie derivative along X. Since, for any fucntion f and a vector Y, $\phi(X)(fY) = f\phi(X)Y$, the Toeplitz operator preserves the tensoriality and is a linear operator. Since $\mathcal{M}$ is a homogeneous Kähler manifold, the Lie derivative preserves the Kähler structure of $\mathcal{M}$ and there is a natural connection ∇ which preserves the Kähler structure. The formulas for the Toeplitz operators in terms of the complexified quantities can be determined from the requirement that both ∇_X and $\mathcal{L}_X$ preserve the Kähler structure. They are

$$\begin{aligned} \phi(H) &= -adH \text{ for } H \in \underline{H}_c, \\ \phi(X_-) &= -\pi_+ \cdot adX_- \text{ for } X_- \in \underline{M}_-, \\ \phi(X_+) &= -(-)^{g(X_+)}\phi(\bar{X}_+)^\dagger \text{ for } X_+ \in \underline{M}_+, \end{aligned} \tag{4.32}$$

where $g(X)$ is the grading of X (0 for L_n and 1 for F_n) and we define $\phi(X_+)$ as the adjoint of $\phi(X_-)$ with respect to the metric. π_+ is the projection operator over $\underline{M}$. We need the phase factor for $\phi(X_+)$, since F_n's are anticommuting quantities.

Since $\mathcal{M}$ is homogeneous, it is enough to calculate the curvature at one point. Then the curvature is, following the expression for a supermanifold,[43] for any vector Z and for any 1-form W on $\mathcal{M}$,

$$\begin{aligned} W \cdot R(X,Y)Z &= (-)^{g(W)\cdot g(Z)} W \cdot \{[\nabla_X, \nabla_Y]_\pm - \nabla_{[X,Y]_\pm}\} Z \\ &= (-)^{g(W)\cdot g(Z)} W \cdot \{[\phi(X), \phi(Y)]_\pm - \phi\left([X,Y]_\pm\right)\} Z, \end{aligned} \tag{4.33}$$

where we have used the fact that for a homogeneous manifold like $\mathcal{M}$ all tangent vectors are Killing vectors so that the Lie derivatives commute with the covariant derivatives.[44] Applied to (4.30) eq.(4.32) gives

$$\begin{aligned} \phi(L_0)L_{-p} &= -[L_0, L_{-p}] = -pL_{-p}, \\ \phi(L_0)F_{-p} &= -[L_0, F_{-p}] = -pF_{-p}, \\ \phi(F_0)L_{-p} &= -[F_0, L_{-p}] = -\frac{1}{2}pF_p, \\ \phi(F_0)F_{-p} &= -[F_0, F_{-p}]_+ = -2L_{-p}, \\ \phi(L_m)L_{-p} &= -\theta(p-m)(p+m)L_{m-p}, \\ \phi(L_m)F_{-p} &= -\theta(p-m)(\frac{1}{2}m+p)F_{m-p}, \\ \phi(F_m)L_{-p} &= -\theta(p-m)(m+\frac{1}{2}p)F_{m-p}, \\ \phi(F_m)F_{-p} &= -2\theta(p-m)L_{m-p}, \\ \phi(L_{-m})L_{-p} &= \frac{(p+2m)(ap^3+bp)}{a(p+m)^3+b(p+m)}L_{-m-p}, \\ \phi(L_{-m})F_{-p} &= \frac{(p+\frac{3}{2}m)(4ap^2+b)}{4a(p+m)^2+b}F_{-p-m}, \\ \phi(F_{-m})L_{-p} &= -\frac{2(ap^3+bp)}{4a(p+m)^2+b}F_{-p-m}, \end{aligned}$$

$$\phi(F_{-m})F_{-p} = -\frac{\frac{1}{2}(p+3m)(4ap^2+b)}{a(p+m)^3+b(p+m)}L_{-m-p}, \tag{4.34}$$

where $p, m > 0$. From eq.(4.33) we get

$$\begin{aligned}
R(L_{-m}, L_n)L_{-p} = \Bigg[&- \theta(p-m)(p+m)^2 \cdot \frac{a(p-m)^3+b(p-m)}{ap^3+bp} \\
&+ \frac{(p+2m)^2(ap^3+bp)}{a(p+m)^3+b(p+m)} - 2mp \Bigg] \delta_{m,n}\delta_p^q L_{-q},
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
R(L_{-m}, L_n)F_{-p} = \Bigg[&\theta(p-m)\left(p+\frac{m}{2}\right)^2 \frac{4a(p-m)^2+b}{4ap^2+b} \\
&- \frac{(p+\frac{3}{2}m)^2(4ap^2+b)}{4a(p+m)^2+b} + 2mp \Bigg] \delta_{m,n}\delta_p^q F_{-q},
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
R(F_{-m}, F_n)L_{-p} = \Bigg[&\theta(p-m)(\frac{1}{2}p+m)^2 \frac{4a(p-m)^2+b}{ap^3+bp} \\
&+ \frac{4(ap^3+bp)}{4a(p+m)^2+b} + 2p \Bigg] \delta_{m,n}\delta_p^q L_{-q},
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
R(F_{-m}, F_{-n})F_{-p} = \Bigg[&- 4\theta(p-m)\frac{a(p-m)^3+b(p-m)}{4ap^2+b} \\
&- \frac{1}{4}\frac{(p+3m)^2(4ap^2+b)}{a(p+m)^3+b(p+m)} - 2p \Bigg] \delta_{m,n}\delta_p^q F_{-q},
\end{aligned} \tag{4.38}$$

where θ is the step function. All other components vanish.

To get the Ricci two-form we have to take trace over positive p of the Riemann curvature tensor. Performing the trace we finds, in the R sector

$$\begin{aligned}
Ric(L_{-m}, L_n) &= -\frac{10}{8}m^3\delta_{m,n}, \\
Ric(F_{-m}, F_n) &= -\frac{10}{2}m^2\delta_{m,n}.
\end{aligned} \tag{4.39}$$

In the NS sector the result is

$$\begin{aligned}
Ric(L_{-m}, L_n) &= (-\frac{10}{8}m^3 + \frac{1}{4}m)\delta_{m,n}, \\
Ric(F_{-r}, F_s) &= (-\frac{10}{2}r^2 + \frac{1}{4})\delta_{r,s},
\end{aligned} \tag{4.40}$$

with r, s half integers. When calculating the *Ricci* form, we find that the infinite series converge remarkably. So we do not need any regularization to calculate the trace.

We combine the result of this section and the curvature of the holomorphic line bundle, eqs.(4.26),(4.27),(4.28),(4.29), to write the curvature of the twisted bundle in the R sector as

$$\begin{aligned} G(L_m, L_{-n}) &= F(L_m, L_{-n}) + Ric(L_{-m}, L_n) \\ &= \left[\frac{1}{8}(d-10)m^3 - 2am\right]\delta_{m,n}, \\ G(F_m, F_{-n}) &= F(F_m, F_{-n}) + Ric(F_{-m}, F_n) \\ &= \left[\frac{1}{2}(d-10)m^2 - 2a\right]\delta_{m,n}. \end{aligned} \tag{4.41}$$

Thus, for $d = 10$ and $a = 0$ we have a gauge invariant description of the open string theory. In passing we note that the vanishing of the ground state energy in the R sector signals the presence of the spacetime supersymmetry. In the NS sector the curvature of the twisted bundle is given by

$$\begin{aligned} G(L_m, L_{-n}) &= \left[\frac{1}{8}(d-10)m^3 - (\frac{1}{8}d - \frac{1}{4} + 2a)m\right]\delta_{m,n}, \\ G(F_r, F_{-s}) &= \left[\frac{1}{2}(d-10)r^2 - (\frac{1}{8}d - \frac{1}{4} + 2a)\right]\delta_{r,s}. \end{aligned} \tag{4.42}$$

Thus, $d = 10$ and $a = -\frac{1}{2}$ gives a reparametrization invariant theory in the NS sector, as expected.

THE CONFORMAL FACTOR AND BRST FORMALISM

As noted by Siegel, [12] the string field should be a functional of the string coordinate, $x^\mu(\sigma)$, and the ghost field, $c(\sigma)$. In this chapter we treat the ghost field geometrically and we modify Bowick and Rajeev's proposal by requiring the closed string field be the Kähler potential of the extended loop space, the space of maps from S^1 to (x^μ, ρ), where S^1 is a circle and ρ is the conformal factor† of a closed string world sheet.

The conformal factor decouples from the S-matrix at the critical dimension in the Polyakov approach. [45] But since the critical dimension comes out as a solution of the equation for the closed string field in the Bowick and Rajeev approach, it is natural to work with the conformal factor from the beginning. In other words we do not require the Weyl invariance at the classical level, but recover it on-shell at the quantum level. To construct the extended loop space, we consider a two dimensional gravity coupled with an open string, $y^\mu(\sigma)$. The action [46] is

$$S = -\frac{1}{2\pi}\int d^2\sigma\sqrt{-g}g^{ab}\partial_a y^\mu \partial_b y_\mu + \frac{1}{8\pi\kappa^2}\int d^2\sigma d^2\sigma' \sqrt{-g(\sigma)}\sqrt{-g(\sigma')}R(\sigma)G(\sigma,\sigma')R(\sigma'), \tag{5.1}$$

where κ is a dimensionless coupling constant, g_{ab} is the world-sheet metric with scalar curvature R, and as usual we have set $\alpha' = 1/2$. The Green's function obeys $\partial_a(\sqrt{-g}g^{ab}\partial_b G(\sigma,\sigma')) = \delta^2(\sigma-\sigma')$. The equation of motion for the metric is

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† In the literature $e^{2\rho}$ is called a confomal factor. But in this chapter for simplicity we call ρ a confomal factor also.

$$T_{ab} = -\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = 0. \tag{5.2}$$

In the conformal gauge, $g = e^{2\tilde{\rho}}dudv$, the trace part of the equation (5.2) gives the equation of motion for the conformal factor, $\partial_u\partial_v\tilde{\rho} = 0$, where we have used the light cone coordinates, u and v. The other components lead to constraint equations which generate the residual reparametrization (the conformal transformation). Comparing with the ordinary covariant string theory, we expect that the conformal factor might play the same role as Faddeev-Popov ghosts due to the gauge fixing in the sense that the excitations of off-diagonal elements of the world-sheet metric are compensated for dynamically by the conformal factor.

The phase space, Γ, of an open string coupled with the two dimensional gravity can be thought of as an extended loop space, if we use a change of variable‡:

$$(x'^{\mu},\rho')(\sigma) = \begin{cases} (p+q')(\sigma), & \text{for } 0\le\sigma\le\pi, \\ (p-q')(-\sigma), & \text{for } -\pi\le\sigma\le 0, \end{cases} \tag{5.3}$$

where $q = (y^{\mu},\tilde{\rho})$ and p is its conjugate variable. Then $(x'^{\mu},\rho')(\sigma)$ forms a loop, since $q'(0) = q'(\pi) = 0$ by the boundary conditions, where $'$ denotes a derivative with respect to σ. In fact the space of variables in the right side of (5.3) is not the phase space of open string but the phase space modded out by the zero modes. Since there is a translational invariance, it is sufficient to consider the phase space modded out by the zero modes. Because of this modding out, the total momentum of the open string, the conjugate variable of the zero mode, is no longer a dynamical variable. We may set the total momentum to be zero, which will not be affected by the dynamics of the open string since the total momentum is a conserved quantity. With this setting, we see that Γ, the space of $(x'^{\mu}(\sigma), \rho'(\sigma))$ is isomorphic to the space of $(x^{\mu}(\sigma), \rho(\sigma))$, which we will call $\tilde{\Gamma}$. The latter will be regarded as the configuration space of a closed string carrying a conformal factor. (Note that $(x^{\mu}(\sigma), \rho(\sigma))$ would not have been periodic, if we had not set the total momentum to be zero.) We note that $\tilde{\Gamma}$ is a loop space. The based loop space, $\Omega = \tilde{\Gamma}/\{(x^{\mu}, \rho)\}$, which is the set of loops with a common point, is a Kähler manifold ^{11} with a complex structure J:

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‡ I would like to thank Prof. Charles Thorn for suggesting the variable in the left side of equation (3).

$$(J(x^{\mu}, \rho))(\sigma) = \sum_{n>0} i(x_n^{\mu}, \rho_n)e^{in\sigma} - \sum_{n<0} i(x_n^{\mu}, \rho_n)e^{in\sigma}, \tag{5.4}$$

where we have mode-expanded $(x^{\mu}, \rho)(\sigma) \in \Omega$ as $(x^{\mu}, \rho)(\sigma) = \sum_{n\neq 0}(x_n^{\mu}, \rho_n)e^{in\sigma}$, and $\bar{x}_m^{\mu} = x_{-m}^{\mu}$ and $\bar{\rho}_m = \rho_{-m}$ because of the reality condition. From now on we will work in this coordinate basis, (x_n^{μ}, ρ_n) for $\mu = 1, 2, \ldots,$ d and $n \in Z/\{0\}$. Since the real line, R, is the Lie algebra of S^1, the based loop space Ω can be thought of as the Lie algebra of the based space, $\tilde{\Omega}$, of a loop group which is a set of maps from $S^1 \mapsto S^1 \times \cdots \times S^1$, where $\cdots$ means a direct product of $d+1$ S^1's. The complex structure is integrable, since (1,0) type vectors form a subalgebra of the Lie algebra of $\tilde{\Omega}$. (The (1,0) type vector, $(e^{in\sigma}, \ldots, e^{in\sigma})\frac{d}{d\sigma}$ with a positive integer n, is an eigenvector of J with an eigenvalue $+i$.) A symplectic two form of Ω is

$$\omega = \sum_{n\neq 0} in(\mathrm{d}x_n^{\mu} \wedge \mathrm{d}x_{-n}^{\nu}\eta_{\mu\nu} + \frac{1}{\kappa^2}\mathrm{d}\rho_n \wedge \mathrm{d}\rho_{-n}), \tag{5.5}$$

and the Kähler potential is

$$K_0 = \sum_{n=1}^{\infty} n(x_n^{\mu}\bar{x}_n^{\nu}\eta_{\mu\nu} + \frac{1}{\kappa^2}\rho_n\bar{\rho}_n) \tag{5.6}$$

We see immediately that the symplectic two form ω in equation (5) gives the same Poisson brackets as the usual canonical Poisson bracket: if we mode-expand $q(\sigma)$ and $p(\sigma)$, $q(\sigma) = q_0 + \alpha_0 \tau + i \sum_{n \neq 0} \alpha_n [(\cos n\sigma)/n]$ and $p(\sigma) = \alpha_0 + \sum_{n \neq 0} \alpha_n \cos n\sigma$, we find that the Poisson brackets $\{x_m, \bar{x}_n\} = \omega^{-1}(dx_m, d\bar{x}_n) = \frac{-i}{m}\delta_{m,n}$ are equivalent to the canonical Poisson brackets $\{\alpha_m, \bar{\alpha}_n\} = im\delta_{m,n}$ or $\{p(\sigma), q(\sigma')\} = \delta(\sigma - \sigma')$ since $imx_m = \alpha_m$.

The classical dynamics of an open string is described by its phase space Γ with a symplectic two form ω. The system carries a set of functions λ_n whose Poisson brackets form a representation of the Lie algebra of vector fields on the circle S^1 with a central extension, $\widehat{\mathcal{D}iff(S^1)}$.

$$\begin{aligned} \{\lambda_m, \lambda_n\} &= \omega^{-1}(\mathrm{d}\lambda_m, \mathrm{d}\lambda_n) \\ &= -i(m-n)\lambda_{m+n} - i\frac{1}{\kappa^2} m^3 \delta_{m+n,0} \end{aligned} \tag{5.7}$$

$$\bar{\lambda}_n = \lambda_{-n},$$

where $\mathrm{d}\lambda_m$ is the exterior derivative of λ_m. Those functions are

$$\lambda_m = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma e^{-im\sigma}\{x'(\sigma)_\mu x'(\sigma)_\nu \eta^{\mu\nu} + \frac{1}{\kappa^2}\rho'^2(\sigma) - \frac{2}{\kappa^2}\rho''(\sigma)\} \tag{5.8}$$

Because of the conformal factor we have introduced, the Poisson bracket has a central term, which implies the system has second class constraints. To quantize the based loop space Ω we apply the geometric quantization techniques.[10] The Hilbert space is a set of sections of the Hermitian line bundle with a metric connection ∇ over Ω, whose curvature is $-i\omega$. So the metric connection is $\nabla = d - iA$, where A is defined by $\omega = dA$. The operators on the Hilbert space, corresponding to the classical observables λ_m, which realize the

Lie algebra, $\widehat{\mathcal{D}iffS^1}$, exist if ω is an integral element of the second cohomology of Ω. The operators§ are

$$r(\lambda_{-m}) = -i\nabla_{X_{\lambda_{-m}}} + \lambda_{-m}, \tag{5.9}$$

where $X_{\lambda_{-m}}$ is the Hamiltonian vector field associated with λ_{-m}, $X_{\lambda_{-m}} = -\omega^{-1}(d\lambda_{-m})$. Explicitly,

$$\begin{aligned} r(\lambda_{-m}) = &- \sum_{p>0}(p+m)(\bar{x}^{\mu}_{p+m}\frac{\partial}{\partial \bar{x}^{\mu}_{p}} + \bar{\rho}_{p+m}\frac{\partial}{\partial \bar{\rho}_{p}}) \\ &- \frac{1}{2}\sum_{p=1}^{m} p(m-p)(\bar{x}^{\mu}_{p}\bar{x}^{\nu}_{m-p}\eta_{\mu\nu} + \frac{1}{\kappa^2}\bar{\rho}_{p}\bar{\rho}_{m-p}), \end{aligned} \tag{5.10}$$

and also for $n = 0$,

$$r(\lambda_0) = -\sum_{p>0} p(\bar{x}^{\mu}_{p}\frac{\partial}{\partial \bar{x}^{\mu}_{p}} + \bar{\rho}_{p}\frac{\partial}{\partial \bar{\rho}_{p}}) + ia, \tag{5.11}$$

where the constant term $(+ia)$ is added due to normal ordering ambiguities. If we had quantized the phase space of the open string in a canonical way, we would have constructed the quantum operator corresponding to a classical observable, a function on the phase space, by replacing the congugate variables in the function by differential operators according to the canonical quantization rule. But in the frame of the geometric quantization that job is done for λ_{-m} by taking the quantum operator as in equation (9). The quantum operator constructed by the above way is the same as the canonical one, as it should be.

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§ Strictly speaking, these operators are prequantization operators, which are defined to act on a wave function (or a section) of both conjugate variables p and q. Upon quantization we have to project out the prequantization operator to get a quantum operator on the quantum Hilbert space. In our case the projection is tantamount to dropping out terms like $\frac{\partial}{\partial x_m}$, $m > 0$. We will use prequantization operators λ_{-m} as quantum operators, assuming that the projection is provided.

According to the uncertainty principle, we have to remove half of the degrees of freedom of sections, because they depend on the phase space. In a Kähler manifold, the natural polarization, $\mathcal{P}$, is such that for any section Ψ,

$$\nabla_v \Psi = 0, \text{ if } v \in \mathcal{P} = \{ \text{ all type (1,0) vectors } \}. \tag{5.12}$$

This polarization is natural in the sense that it is equivalent to the holomorphic condition. So the quantum Hilbert space is a set of sections of the holomorphic, Hermitian line bundle. In a local coordinate chart if we set $\Psi = e^{-K_0/2}\tilde{\Psi}$, where K_0 is the Kähler potential, the polarization, $\nabla_v \Psi = 0$ implies $\partial_v \tilde{\Psi} = 0$. In other words, $\tilde{\Psi}$ is a holomorphic function of $\bar{x}_\mu$ and $\bar{p}_n$. Among operators in equation (5.10), they preserve the polarization condition, if $m \leq 0$. Since we want the inner product of the Hermitian bundle to be preserved by the action of $r(\lambda_m)$, for $m > 0$, we take

$$r(\lambda_m) = -r^\dagger(\lambda_{-m}), \tag{5.13}$$

where † is a Hermitian conjugate with respect to the inner product. The space of all complex structures of Ω is $\mathcal{D}iff(S^1)/S^1$. Physical string amplitudes should be independent of the way in which strings are parametrized. For the Kähler manifold Ω, each complex structure represents a reparametrization up to S^1. The manifold $\mathcal{M}$=$\mathcal{D}iff(S^1)/S^1$ is a Kähler manifold with a Kähler two form

$$\tilde{\omega}(L_m, L_n) = (am^3 + bm)\delta_{m+n,0}, \tag{5.14}$$

where L_m, L_n are the elements of the Virasoro algebra and a and b are either (1) a=0 and $b \neq 0$ or (2) $a \neq 0$ and $-b/a$ is not equal to the square of a nonzero integer. Since $\mathcal{M}$ is also a coadjoint orbit of the Virasoro group, it is important to consider a holomorphic Hermtian line bundle over $\mathcal{M}$ taking the fibre as the quantum Hilbert space of the open string. $\mathcal{M}$ is quantizable and the sections are unitary representations of the Virasoro group, [47,48] because $\tilde{\omega}$ is an integral element of the second cohomology of the Virasoro group, $\mathcal{V}$. (The generator of $H^2(\mathcal{V},C)$ is $\frac{1}{12}m(m^2-1)\delta_{m+n,0}$.[38]) Since the curvature does not depend on a choice of the connection, we take $\nabla_{L_m} = \mathcal{L}_{L_m} + r(\lambda_m)$, which preserves the inner product because of the choice (5.13). $\mathcal{L}_{L_m}$ is a Lie derivative along L_m. Then the curvature two form of the bundle is

$$\begin{aligned} R(L_m, L_n) &= [\nabla_{L_m}, \nabla_{L_n}] - \nabla_{[L_m,L_n]} \\ &= [r(\lambda_m), r(\lambda_n)] - r(\{\lambda_m, \lambda_n\}) \\ &= (\frac{\mathrm{d}+1}{12} + \frac{1}{\kappa^2})m^3\delta_{m+n,0} - (2a + \frac{\mathrm{d}+1}{12})m\delta_{m+n,0}, \end{aligned} \tag{5.15}$$

which is equal to $\tilde{\omega}$, the Kähler two form of $\mathcal{M}$, as we want. The curvature of the line bundle is required to be zero to have a covariantly constant section Ψ, which we interpret as a vacuum of the Hilbert space. The covariantly constant section is $e^{-K_0/2}$ up to overall normalization. Apart from the measure of the line bundle, the section has no excitations at all. Other states in the Hilbert space are generated, in a local coordinate chart, by holomorphic functions, $\Phi(\bar{x}_n, \bar{\rho}_n) = f(\bar{x}_n, \bar{\rho}_n)e^{-K_0/2}$, where $f(\bar{x}_n, \bar{\rho}_n)$ is a holomorphic function of $\bar{x}_n$ and $\bar{\rho}_n$.

The critical dimension at which the curvature is zero depends on the two dimensional gravitational coupling constant κ^2. But it turns out that κ^2 should be -4/9. Therefore, we get the critical dimension, $d = 26$ and the vacuum energy is $a = -9/8$. The Hamiltonian of the open string coupled with the two dimensional gravity is

$$H = \int \mathrm{d}\sigma^a \xi^b T_{ab}, \tag{5.16}$$

where T_{ab} is the stress energy tensor and ξ^b is the time-like Killing vector, $\xi = e^{-\rho(u)}\frac{\partial}{\partial u} + e^{-\rho(v)}\frac{\partial}{\partial v}$. The Hamiltonian is classically invariant under reparametrizations. We require that it also be invariant quantum mechanically under reparametrizations. The Killing vector is modified as $\xi^u =: e^{-\alpha\rho(u)}:$ to have the conformal weight -1 quantum mechanically. The conformal Killing vector is anticommuting if $\alpha^2 = (2n+1)\kappa^2$, $n \in Z$.[49] Assuming ξ is a dynamical field, we find that its conjugate field is a quadratic differential, $: e^{-\beta\rho(u)} : du^2 + : e^{-\beta\rho(v)} : dv^2$, since the Killing vector satisfies a first order equation, $\nabla_a \xi_b + \nabla_b \xi_a = 0$. Since the vacuum correlation function is $\langle \rho(u)\rho(\tilde{u})\rangle = -\kappa^2 \log(u - \tilde{u})$, we find that the unique solution is $\alpha = -\beta = 3/2$ and $\kappa^2 = -4/9$, taking $n = 0$. We see that the Killing vector and its conjugate field are all anticommuting fields. Furthermore, the Hamiltonian is nilpotent in the critical dimension. That κ^2 is negative implies that the conformal factors generate negative norm states on the string world-sheet. Identifying $\phi = i\tilde{\rho}/\kappa$, as noted by Horowitz and Strominger[50], we see that ϕ is the bosonized BRST ghost and the Hamiltonian is the BRST charge. The prequantization operator, corresponding to the classical Hamiltonian, is

$$r(H) = -i\nabla_{X_H} + H, \tag{5.17}$$

where X_H is the Hamiltonian vector field associated with H, $X_H = -\omega^{-1}(dH)$. But $r(H)$ does not preserve the polarization, $\mathcal{P}$, since X_H is not holomorphic. Hence the quantum operator, H_{op}, can not be constructed by projecting out the prequantization operator, $r(H)$. To get H_{op} we have to use other methods or we have to introduce other basis, where X_H is holomorphic. In a local coordinate chart, H_{op} can be constructed in the canonical way, namely by replacing x_m and ρ_m by $-\frac{1}{m}\frac{\partial}{\partial \bar{x}_m}$ and $-\frac{\kappa^2}{m}\frac{\partial}{\partial \bar{\rho}_m}$, $m > 0$, respectively, which results in $H_{op} =$

Q_{BRST}, the first-quantized BRST operator. (For a trivial bundle, where ω is exact, this construction covers whole bundle.) With guidance of the string field theory proposed on the basis of BRST formalism[51] we propose that the (free) action for the open string field is

$$\begin{aligned} S &= \langle \Psi | H_{op} | \Psi \rangle \\ &= \int \prod_{n=-\infty}^{\infty} dx_n d\rho_n \bar{\Psi} H_{op} \Psi. \end{aligned} \tag{5.18}$$

Then the field equation for the open string is $H_{op}\Psi = 0$, which is nothing but the BRST equation. If we treat the closed string field as a background field, the open string field lives in the critical dimension, where H_{op} is nilpotent. Then we see that the action is invariant under a transformation $\Psi \mapsto \Psi + H_{op}\Lambda$, where Λ is an arbitrary section. This gauge invariance is crucial to maintain the unitarity in open string field theory. It is expected that we get only free open string field equation, since what we did is to lift the first quantized equation to the field equation by taking the wave function as the string field, and, as we know, the first quantized string theory is a theory of one string theory, while the string field theory is a theory of many string theory. So, in a string field theory, the interaction is natural and the basic string interaction is a cubic (and nonlocal in the loop space) interaction, which describes the joining and splitting of the string, and other complicate interaction of string can be patched with this basic cubic worldsheet configuration. It is straightforward to include interaction terms as in the usual open string field theory.[52] The action for the Witten's string field theory contains a measure for the integration, which is propotional to e^{-K}, where K is the Kähler potential of the holomorphic bundle and is interpreted as the closed string field in the Bowick and Rajeev proposal. So we see that the open string field propagates under the background of the closed string field, which will be the loop space correction if the Bowick and Rajeev approach is relevant to the string field theory.

In conclusion, by introducing a dynamical conformal factor we see that the Faddeev-Popov ghosts enter on the same footing as the x^μ fields in the geometrical formulation following Bowick and Rajeev, where the conformal factor is identified with the bosonized ghost. The closed string field is proposed to be the Kähler potential of the extended loop space, which is a function of x^μ and the conformal factor ρ. An open string field is a section of a holomorphic, hermitian line bundle over $\mathcal{D}iff(S^1)/S^1$, whose curvature is zero as the equation of motion for the closed string field. The BRST formalism for the open string field is derived in Bowick and Rajeev's approach.

It will be interesting to extend our approach to the supersymmetric case. For an example, one may try the supersymmetric Liouville theory, since it is a natural supersymmetric extension of our theory. But it turns out to be that the supersymmetric Liouville theory does not have enough degrees of freedom to account the superghost fields. To have enough degrees of freedom, one may look at two dimensional complex supermanifold.

APPENDIX

Q-BALLS IN SUPERFLUID ^3He

Below 100 mK the liquid ^3He is well described by Landau's Fermi liquid theory. According to the theory, to describe the liquid ^3He, we not only introduce the concept of quasiparticles, but also consider the phenomenological effective interactions between quasiparticles (= the fluctuation exchange interactions). [53] The effective interaction between quasiparticles becomes strong enough below about 3 mK to form vacuum condensates, the so-called Cooper pairs. The order parameter of the Cooper pair formed in the l=1, spin triplet state, can be parametrized as $\Delta_{ai} =< \psi \sigma_2 \sigma_a \overset{\leftrightarrow}{\nabla}_i \psi >$, where ψ is the wave fuction of the quasiparticle and σ_a, a=1,2,3, are Pauli matrices.

The symmetry of the hamiltonian of the liquid ^3He is $SO(3)_S \times SO(3)_L \times U(1)_I$, if we neglect the dipole interaction; the $U(1)_I$ symmetry comes from the invariance under phase rotations of the quasiparticle.

For temperatures near the critical temperature we may use the Landau-Ginzburg free energy to describe the superfluid ^3He. In the Landau-Ginzburg region, the minimum free energy configuration for the Cooper pair is $\Delta_{ai}(\vec{n}) \sim e^{i\phi} \cdot R_{ji}(\vec{n}, \theta)\delta_{aj}$, where $\vec{n}$ is a unit vector in momentum space and R_{ji} is the orthogonal matrix for rotations by θ about the axix $\vec{n}$. This configuration is known as that of the B phase. The simultaneous rotation of the spin space and ordinary space leaves $\Delta_{ai}(\vec{n})$ invariant. Therefore the unbroken symmetry is $SO(3)_J$, with $\vec{J} = \vec{L} + \vec{S}$. When Cooper pairs are formed, the interaction between quasiparticles is modified by so-called feedback effects. As a consequence of the feedback effects for temperatures between T_C and T_{AB} the minimum free energy configuration is $\Delta_{ai} \sim \hat{z}_a(\hat{x}_i + i\hat{y}_i)$, which is called the A phase. The unbroken symmetry of the A phase is $U(1)_{S_z} \times U(1)_{\tilde{L}_z}$, where $\tilde{L}_z = I - L_z$.

The A-B transition is known to be a first order phase transition. Supercooling is observed for the A-B transition. [54] Hakonen *et al.* held the ^3He-A phase stable below T_{AB} for up to 8 hours.[55] There have been several attempts to explain the A-B transition. [56] But no mechanism has yet been proposed to get a reasonable transition rate. One of them uses a *critical bubble*[57] which is a kind of distorted order parameter,

$$\Delta_{ai}(|\vec{x}|) = \begin{cases} \Delta_{ai}(B), & \text{as } |\vec{x}| \text{ goes to } 0, \\ \Delta_{ai}(A), & \text{as } |\vec{x}| \text{ goes to } \infty. \end{cases} \tag{A.1}$$

Since $F_A > F_B$ for $T < T_{AB}$, the surface energy is compensated by the gain in the free energy so that it can be energetically favorable to form a *critical bubble* in the supercooled ^3He-A phase. But the nucleation rate for a *critical bubble* turns out to be exponentially small, because it occurs through quantum tunneling or thermal fluctuations.[58] An external disturbance[59] or a seed is needed in the supercooled ^3He-A phase to trigger nucleation.

For temperatures higher than T_{AB} the chance that the *critical bubble* exists is extremely small. But according to Coleman[60] still for $T > T_{AB}$ there exists a new stable configuration, so-called Q-ball. This nontopological soliton is a minimum energy configuration for a given charge Q. (In our case Q will be S_z or $\tilde{L}_z$. But we will write generically Q for S_z or $\tilde{L}_z$, following Coleman.) Note that the *critical bubble* has no spin or angular momentum, since it is a static configuration. The Q-ball configuration is quite similar to that of the critical bubble except that the Q-ball has a time dependent phase. The Q-ball can be described as rotating in spin space or as having its phase change while it rotates in ordinary space. So it has spin or $\tilde{L}_z$, whereas the *critical bubble* has no charge. Indeed the Q-ball is stable because of charge conservation.

Among the 18 collective modes of the ^3He-A phase there are 5 Nambu-Goldstone modes which are small excitations along the flat directions of the A phase vacuum in the free energy. The other 13 modes are small oscillations around the A phase vacuum and feel potential barriers between A and B phases. [61] We note that these 13 modes are therefore responsible for the A-B transition. As a simplifying step let us consider only the pseudospin mode which has $S_z = 1$ and $\tilde{L}_z = 0$ under $U(1)_{S_z} \times U(1)_{\tilde{L}_z}$. [62] We present the analysis for the pseudospin mode and will just make a comment about other modes later, since the analysis can be easily applied to other modes, but does not change the qualitative overall behavior.

The most general Lagrangian for the pseudospin mode ϕ can be written as

$$L = \frac{1}{v^2}\dot{\phi}^*\dot{\phi} - \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - U(\phi) + L', \tag{A.2}$$

where $U(\phi)$ is the self interaction of pseudospin modes and L' denotes the interaction with other modes and quasiparticles. Because of L' the pseudospin mode dissipates. Indeed it decays into one quasiparticle and one quasihole. For the pseudospin mode the causality propagation speed, v, is the Fermi velocity, $v_F \simeq 10^3 cm/sec$, up to a factor of order 1 coming from the renormalization effects. [63] From now on we take v as v_F, since in this paper our accuracy is less than 10%, because of the uncertainty in the parameters of the Lagrangian. The potential energy $U(\phi)$ for pseudospin modes has a global minimum at $\phi = 0$. It has also a local minimum near $\phi = \phi_B$, since there is a potential barrier in the free energy between the A phase configuration and the B phase configuration. $\phi_B = \langle\Omega_B|\phi|\Omega_B\rangle$ measures the overlap in the wave function between the pseudospin mode and the B phase ground state, $|\Omega_B\rangle$. The potential energy can be written as

$$U(\phi) = \alpha_1\phi^*\phi + \alpha_2(\phi^*\phi)^2 + \alpha_3(\phi^*\phi)^3 + \ldots + \beta_1(\vec{\nabla}\phi^* \cdot \vec{\nabla}\phi)^2 + \ldots$$
$$\simeq (\frac{\mu^2}{\phi_B^4})\tilde{\phi}^2(\tilde{\phi}^2 - \phi_B^2)^2 + \lambda(T)\tilde{\phi}^4, \qquad \text{(A.3)}$$

where we neglect higher order terms in the potential energy. This approximation is valid in the Landau-Ginzburg region. $\tilde{\phi}$ is the magnitude of the pseudospin mode ϕ. $\lambda(T)$ is proportional to the free energy difference, [64] $(F_B - F_A)/F_B \simeq 0.17(T/T_{AB} - 1)$. Certainly the frequency of the pseudospin mode, $\omega_S = \mu v_F$ †, depends on the temperature, T. But the dependence is negligible, compared with $\lambda(T)$. Now let us find a solution of the equation of motion for the pseudospin mode, which is of a Q-ball configuration;

$$\phi(\vec{x}, t) = \tilde{\phi}(r)e^{i\omega t}. \qquad \text{(A.4)}$$

$\tilde{\phi}(r)$ decreases monotonically from $\tilde{\phi} = \phi_B$ to $\tilde{\phi} = 0$ as $r = |\vec{x}|$ goes from 0 to ∞. (4) is shown to be the solution for some ω, if $\omega_0 < \omega < \omega_S$,[60] where $\omega_0 = v_F\phi_B\sqrt{\lambda(T)}$. The frequency of the pseudospin mode is $\omega_S \sim \Delta(0)/\hbar$, [61] where

$$\Delta(0) = \lim_{T\to 0}\Delta(T) \simeq 2.03k_BT_C \qquad \text{(A.5)}$$

is the gap energy at $T = 0$. Then the frequency of a Q-ball is at most $\omega = \omega_S \simeq v_F/\xi_0$, if we take $\Delta(0) \simeq \hbar v_F/\xi_0$, which is as large as the inverse of the characteristic adjustment time $\hbar/\Delta(0) \simeq \xi_0/v_F$ of the Cooper pairs. Therefore the Q-ball solution is consistent with the assumption of the Landau-Ginzburg free energy. For the Q-ball to be easily excited from the ground state of the A phase, it should be a minimum energy configuration. As an approximation we take $\tilde{\phi}(r)$ of the Q-ball to be

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† The dimensions of the relevant quantities are the following;

$\dim[\phi/\sqrt{\hbar}] = [length{\cdot}time]^{-1/2}$, $\dim[Q/\hbar] = 0$, $\dim[\omega] = \dim[\omega_0] = [time]^{-1}$, $\dim[\mu] = [length]^{-1}$, and $\dim[\hbar\lambda(T)] = [length]^{-1}[time]$.

$$\tilde{\phi}(r) = \begin{cases} \phi_B, & \text{if } r \leq R, \\ 0, & \text{otherwise,} \end{cases} \tag{A.6}$$

which is a mild approximation as long as the radius of the Q-ball, R, is large compared with the width of the surface of the Q-ball, which is about $1/\mu$. The energy of the Q-ball is

$$\begin{aligned} E &= \int d^3x \{ \frac{1}{v_F^2} \dot{\phi}^* \dot{\phi} + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + U(\phi) \} \\ &= (\omega/v_F)^2 \phi_B^2 V + U(\phi_B) V + 4\pi R^2 \sigma, \end{aligned} \tag{A.7}$$

where σ is the surface tension, and V is the volume of the Q-ball. In the same approximation the charge of the Q-ball is

$$\begin{aligned} Q &= \frac{i}{v_F^2} \int d^3x (\dot{\phi}^* \phi - \phi^* \dot{\phi}) \\ &= 2(\omega/v_F^2) \phi_B^2 V. \end{aligned} \tag{A.8}$$

For fixed charge Q, if we neglect the surface energy, E has its minimum at

$$V = \frac{Q v_F}{2\sqrt{\phi_B^2 U(\phi_B)}}. \tag{A.9}$$

Therefore the Q-ball is a minimum energy configuration with frequency $\omega = v_F \sqrt{U(\phi_B)/\phi_B^2}$, energy $E = \omega Q$, and radius $R = (3/4\pi)^{1/3} \cdot (Q v_F^2/\omega)^{1/3} \cdot \phi_B^{-2/3}$. The surface energy is, from the equation (A.7),

$$\begin{aligned} E_S &= \int d^3x \{ U(\phi) + (\frac{d\tilde{\phi}}{dr})^2 \} - U(\phi_B) V \\ &= \int d^3x \{ U(\phi) + (\frac{d\tilde{\phi}}{dr})^2 - \frac{\omega^2}{v_F^2} \tilde{\phi}^2 \}, \end{aligned} \tag{A.10}$$

where we used $U(\phi_B)V = \frac{\omega^2}{v_F^2}\phi_B^2 V = \frac{\omega^2}{v_F^2}\int \tilde{\phi}^2 d^3x$. Using the equation of motion and the fact that the integrand has a peak at $r = R$, and approximating $\omega \simeq \omega_0$, we get

$$\begin{aligned} E_S &\simeq 4\pi R^2 \int_0^{\tilde{\phi}_B} 2\sqrt{U(\phi) - \omega_0^2\tilde{\phi}^2}\, d\tilde{\phi} \\ &\simeq 4\pi R^2 \cdot \mu\phi_B^2. \end{aligned} \tag{A.11}$$

Therefore the ratio is

$$\begin{aligned} \frac{E_V}{E_S} &\simeq \frac{2}{3}R \cdot \frac{\omega_0^2}{\mu v_F^2} \\ &\sim \frac{2}{3}(T/T_{AB} - 1)R/\xi_0. \end{aligned} \tag{A.12}$$

We see that the condition $R \gg \xi_0$ justifies the neglect of the surface energy as long as T is not very near T_{AB}. In this paper we take the following approximations on dimensional grounds:

$$\begin{aligned} \phi_B/\sqrt{\hbar} &\simeq \frac{\sqrt{v_F}}{\xi_0}, \\ \hbar\lambda(T)\phi_B^2 &\simeq \frac{1}{\xi_0^2}|T/T_{AB} - 1|, \\ \mu &\simeq \frac{1}{\xi_0}, \\ \omega_0 &\simeq \frac{v_F}{\xi_0}|T/T_{AB} - 1|^{1/2}, \end{aligned} \tag{A.13}$$

where $\simeq$ means equality up to an order of magnitude. (Note that v_F and ξ_0 are the only intrinsic parameters in the superfluid ${}^3He$.) Once thermally excited pseudospin modes are correlated over a region of size $R \gg \xi_0$, they should form Q-balls, since it is energetically favorable ($Q\omega_0 \ll Q\omega_S$ near T_{AB}). To calculate the formation rate, it is enough to consider only the correlation probability as in usual solitons. This is one of the salient differences between the Q-ball and a *critical bubble*. In other words, forming a Q-ball is not a tunneling process at all, but a kind of kinematical process. Indeed it is a thermal process toward a configuration of low energy and of low entropy. (The Q-ball is a configuration of very low entropy, because the region of size $R \gg \xi_0$ is correlated.) The probability to be correlated over a region of size R is $e^{-R/\xi}$ ‡, where ξ is the correlation length at T. Near T_C the correlation length is $\xi \simeq \xi_0(1 - T/T_C)^{-1/2}$. The formation rate per unit time is

$$F.R. = \frac{1}{R/v_F} \cdot \frac{\Omega}{\frac{4\pi}{3}R^3} \cdot e^{-R/\xi}, \tag{A.14}$$

where R/v_F is the time for the Q-ball to be correlated, and Ω is the volume of the superfluid ^3He. The second factor in $F.R.$ is just a geometrical prefactor. Taking $\Omega = 1cm^3$, $v_F = 10^3 cm/sec$, and $\xi_0 = 500\mathring{A}$, we get

$$F.R. \simeq 3.8 \times 10^{23} sec^{-1} \cdot (\frac{R}{\xi_0})^{-4} e^{-R/\xi}. \tag{A.15}$$

Since the rate depends on R/ξ exponentially, it has a fair amount of uncertainty. It varies from 1 per second to 1 per 100 years, as R/ξ varies from 30 to 100, assuming $\xi \simeq 10\xi_0$. But once a Q-ball is formed, it will not decay into pseudospin modes because of the energy barrier and the conserved charge Q. But it decays into other modes and quasiparticles, since the pseudospin mode itself dissipates.

Since the charge of a Q-ball is conserved for $T > T_{AB}$, as we lower the temperature T with a rate $\dot{T}$ the Q-ball grows with a velocity

$$\dot{R} = \frac{1}{6}\frac{\dot{T}}{T_{AB}} R_0 (\frac{T_0}{T_{AB}} - 1)^{1/6} (\frac{T}{T_{AB}} - 1)^{-7/6}$$

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‡ If we divide R into small intervals of length $\epsilon \ll R$, then for each ϵ the probability to be correlated over ϵ is $1 - \frac{\epsilon}{\xi}$. Therfore, the probability to be correlated over R is

$$w = \lim_{\epsilon \to 0} (1 - \frac{\epsilon}{\xi})^{R/\epsilon} = e^{-R/\xi}$$

$$\simeq 4.0 \times 10^{-6} cm/sec \cdot (\frac{\dot{T}}{1\mu K/sec}) \cdot (\frac{R_0}{0.15cm}) \cdot (\frac{T}{T_{AB}} - 1)^{-7/6}$$
$$\geq 1.3 \times 10^{-2} cm/sec, \qquad (A.16)$$

where the first step is obtained by taking a derivative with respect to time of the equation (8) and the temperature cooling rate is taken to be $\dot{T} = 1\mu K/sec$. R_0 is the radius of the Q-ball formed at T_0, and we take $T_0 = 1.001 T_{AB}$ and $R_0 = 3 \times 10^4 \xi_0$, because we will see that at this radius and temperature the growth rate of the Q-ball is larger than the evaporation rate. We see that the velocity is small unless T is very near T_{AB}.

The Q-ball decays into other modes, because of its coupling with other modes. In order for the Q-ball to decay, the energy of those modes should be less than one half of the energy of the Q-ball per unit charge, $\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar v_F/\xi_0\sqrt{T/T_{AB}-1} = \frac{1}{2}\Delta(0)\sqrt{T/T_{AB}-1}$. Therefore near T_{AB} Q-balls can decay only into Nambu-Goldstone modes † or massless fermionic excitations. If we treat the Q-ball as a classical background field, we can calculate the decay rate by knowing the vacuum to vacuum amplitude. It turns out that the magnitude of the amplitude is 1, because the propagator of the Nambu-Goldstone mode has no pole at the imaginary value of its energy. So the norm of the Q-ball is preserved. Hence Q-balls do not decay into Nambu-Goldstone modes in the inside of Q-balls. Instead they evaporate into pairs of Nambu-Goldstone modes. The pair production rate per unit area per unit time is the expectation value of the flux of Nambu-Goldstone modes produced at the surface of the Q-ball. According to Cohen *et al.* [65], the velocity of recession of the Q-ball surface is

$$\frac{dR}{dt} \leq \frac{\hbar\omega_0^2}{384\pi^2\phi_B^2}$$

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† I would like to thank Professor S. Coleman for pointing out this possibility.

$$\simeq 2.6 \times 10^{-4}(\frac{T}{T_{AB}} - 1)v_F, \qquad (A.17)$$

which is about $2.6 \times 10^{-4} cm/sec$ for $T = 1.001T_{AB}$. The calculation for the decay rate of the Q-ball into massless fermionic excitations is similar. Because of the Fermi pressure of the Dirac sea of massless fermionic excitations[65], the decay occurs only on the surface of the Q-ball.§ The absolute bound on the velocity of the recession due to the evaporation into massless fermionic excitations is of the same order as equation (A.17). Unless Q-balls are formed very near T_{AB}, say $T = 1.001T_{AB}$, they evaporate easily. Only Q-balls formed very near T_{AB} grow. When a Q-ball grows enough, the surface energy is no longer negligible. $E_V = Q\omega_0 \simeq Qv_F/\xi_0\sqrt{T/T_{AB} - 1}$ goes to 0, while $E_S \simeq 4\pi R^2 \mu \phi_B^2$ becomes large, as T goes to T_{AB}. Therfore the Q-ball configuration considered in this paper is no longer the minimum energy configuration. It will evolve into another configuration, which minimizes the energy including the surface energy.[66] (We will call the surface-energy-dominated Q-ball a type II Q-ball, while refering to the one considered earlier in this paper as type I Q-ball.) But it is more likely that the type II of radius $R \gg \xi_0$ (*e.g.* $R = 10^3\xi_0$) is formed near T_{AB}, instead of evolving from the type I Q-ball. The formation rate for type II Q-ball also depends on the radius. So the formation rate is, from equation (A.15) $F.R. \simeq 1.8 \times 10^{-3}/sec$ if we take $R = 10^3\xi_0$ and $\xi = 30\xi_0$.

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§ For finite temperature we have to consider the possibility that the Q-ball decays into massless fermionic excitations because the Dirac sea is partially occupied near the Fermi temperature, T_F, due to thermal fluctuations. The fraction is of the order of $\frac{T}{T_F}$. The amplitude is proportional to $(\frac{T}{T_F})^2$, because the Q-ball decays into a pair of massless fermionic modes. Therefore the decay rate is proportional to $(\frac{T}{T_F})^4$ and the recession velocity due to this decay is about $(\frac{T}{T_F})^4 \cdot v_F \simeq 10^{-9} cm/sec$, which is somewhat smaller than the velocity due to the evaporation.

The energy of the Q-ball is, from equations (A.11) and (A.12), using the expression for the charge Q,

$$E = \frac{3Q^2v_F^2}{16\pi R^3\phi_B^2} + \lambda(T)\phi_B^2\frac{4\pi}{3}R^3 + 4\pi R^2\mu\phi_B^2. \tag{A.18}$$

The energy has its minimum at $R = R_m$, which satisfies

$$\lambda(T)\phi_B^6R_m^6 + 2\mu\phi_B^4R_m^5 - (\frac{9}{64\pi^2})v_F^2Q^2 = 0. \tag{A.19}$$

At T_{AB}, the radius of Q-ball is $R_m(T_{AB}) = (\frac{9}{128\pi^2})^{1/5}(\frac{Q}{\hbar})^{2/5} \simeq 3.7 \times 10^3\xi_0$ for $Q = 10^{10}\hbar$. Taking a derivative of the equation (A.19) with respect to temperature, we get

$$\frac{dR}{dT} = -\frac{\phi_B^2R^2\frac{d\lambda}{dT}}{6\lambda(T)\phi_B^2R + 10\mu\phi_B^2}. \tag{A.20}$$

Integrating the equation (A.20), we obtain

$$\frac{T}{T_{AB}} = 1 - \frac{2}{(R/\xi_0)} + \frac{c}{(R/\xi_0)^6}, \tag{A.21}$$

where $c = (\frac{3^{12/5}}{2^{42/5}\pi^{12/5}})(\frac{Q}{\hbar})^{12/5}\{1 - \frac{2^{12/5}\pi^{2/5}}{3^{2/5}}(\frac{Q}{\hbar})^{-2/5}\}$, and we have used the approximation (A.13). Since $\lambda(T)\phi_B^2 \simeq \frac{1}{\xi_0^2}\sqrt{T/T_{AB} - 1}$, the radius of the Q-ball increases as the temperature decreases. If we decrease T below T_{AB}, the radius still gets large and $\frac{dR}{dT}$ is singular at T_*, the temperature at which the denominator of the equation (A.20) is zero:

$$\begin{aligned}\lambda(T_*)\phi_B^2 &= -\frac{5\mu}{3R}\\ &= -(\frac{2^6 5^5\pi^2}{3^8})^{1/5}(\frac{\mu^6\phi_B^4}{v_F^2})^{1/5}Q^{-2/5},\end{aligned} \tag{A.22}$$

where the equation (A.19) is used at the second stage. At T_* the Q-ball becomes unstable and it grows indefinitely. T_* is the onset temperature up to which the ^3He-A phase supercools and then the Q-ball grows classically to be the ^3He-B phase vacuum. The radius of the Q-ball is a function of T. The classical equation for the motion of the Q-ball is simply

$$M\ddot{R} + \Gamma \dot{R} = F, \tag{A.23}$$

where M is the mass of the Q-ball and Γ is the friction coefficient due to the scattering of quasiparticles on the surface of the Q-ball.[67] F is the force per unit area. If we increase the radius of the Q-ball, it will gain energy which will be converted into kinetic energy of the surface element of the Q-ball. The force per unit area is

$$\begin{aligned} F &= -\frac{1}{4\pi R^2}\frac{dE}{dR} \\ &= \frac{9Q^2 v_F^2}{16\pi^2 R^4 \phi_B^2} - \lambda(T)\phi_B^4 - \frac{2\mu\phi_B^2}{R}. \end{aligned} \tag{A.24}$$

If the Q-ball grows enough, $\dot{R}$ approaches to the terminal velocity $v_f = F/\Gamma$. For very large R, $F \simeq -\lambda(T)\phi_B^4 = U(0) - U(\phi_B)$, which is the free energy difference per unit volume between two phases. Yip and Leggett calculated Γ when $R \to \infty$.[67] We note that in this limit we recover their result, which agrees to a good extent with the experiment by Buchanan *et al.*[68] For finite R we will get some corrections due to the finite size of the Q-ball and the time dependence of the Q-ball. But they will be negligible, because the radius of the Q-ball is sufficiently large when its velocity approaches the terminal velocity. The radius of the Q-ball at T_* depends on the charge of the Q-ball. From equation (A.21), $R(T_*) = (3c)^{1/5}\xi_0 = 0.43\xi_0(Q/\hbar)^{12/25}\{1 - 5.38(Q/\hbar)^{-2/5}\}^{1/5}$, where c is the integration constant in equation (A.21). For $Q = 10^{10}\hbar$, $R(T_*) \simeq 2.7 \times 10^4 \xi_0$. Below T_{AB} we can treat the metastable ^3He-A phase vacuum as the vacuum of the theory. So the charge Q is still well-defined. If the radius of the Q-ball goes to ∞, the frequency of the Q-ball goes to zero since $Q = \frac{8\pi}{3}R^3\omega\phi_B^2$ is conserved. Therefore when the Q-ball occupies the whole space, we achieve the phase transition with a fair amount of supercooling, which is larger than $T_{AB} - T_*$. For a probable value of Q, $10^{10}\hbar$, $T_{AB} - T_* \simeq 0.001 mK$. See table 1 for other values. So far we have analyzed the pseudospin mode only. The analysis for other massive modes will be similar. It is also likely that Q-balls with charge S_z and $\tilde{L}_z$ are formed near T_{AB} out of thermally excited modes other than the pseudospin mode. Since Kaul and Kleint discovered a minimum free energy path between the A phase vacuum and the B phase vacumm,[64] the Q-ball configuration may be of the following form;

$$\Delta(r,t)_{ai} = O_{S_z}(\omega_1 t)_{ab} O_{\tilde{L}_z}(\omega_2 t)_{ij} \Delta(r)_{bj}, \tag{A.25}$$

where $\Delta(r)_{bj}$ is the minimum path and $O_{S_z}, O_{\tilde{L}_z}$ are the rotational matrices in spin space and $\tilde{L}_z$ space respectively. This configuration will be the minimum energy configuration for the fixed charge if ω_1 and ω_2 are much smaller than the minimum of the frequencies of the massive collective modes, and the radius of this configuration is large enough ($R \gg \xi_0$). But our analysis does not depends on the detail of the path as long as it is the minimum energy configuration. So the behavior of the configuration, equation (A.25), will be similar to that in equation (4). In other words the supercooling amount and the velocity of the surface of the Q-ball can be calculated once we know the charge of the probable Q-ball. They will be of same order as those of the Q-ball in the pseudospin modes.

Since we can predict only the order of the magnitude for the quantities like the supercooling amount, the velocity, *etc.*, we do not know whether the Q-ball is really the mechanism for the ^3He A-B transition. But we believe that Q-balls provide the most likely scenario for the observed phase transition, because they show the features of a first order phase transition. Near $T > T_{AB}$, a surface-energy dominated Q-ball is formed and evolves till $T_* < T_{AB}$. At T_* it becomes unstable and grows classically with a terminal velocity. If the Q-ball grows enough, eventually it occupies the whole space, so we achieve the phase transition. Because of the uncertainty in the Lagrangian, we do not know if the Q-ball is formed at a sufficient rate. But it is quite likely that the Q-ball will act as a seed for the *critical bubble* to be nucleated easily. Also as a soliton the Q-ball is a very interesting object in the ^3He. The Q-ball can change the magnetic properties of the ^3He-A phase, because it has huge spin and/or angular momentum. If the ^3He-A phase shows an anomalous magnetic behavior near T_{AB}, it may be indirect evidence for Q-balls. Further studies in this direction are needed.

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BIOGRAPHICAL SKETCH

Deog Ki Hong was born the first son of Sung Ahn Hong and Jung Hyun Kang in Jejudo, Korea, on November 23, 1959 (as lunar calendar). He was raised and educated in Pusan, Korea. After high school, he went to Seoul National University to major in physics in 1977 and was awarded a B.S. in 1981. He joined the graduate school of his home university to continue his career as a theoretical Physicist in 1981 and was awarded a M.S. degree in 1983. His thesis topic for M.S. was grand unified theories, paricularly $SO(14)$ model. Shortly after graduation he joined army for six months, then taught mathematical physics at Inha University in Korea as a part-time lecturer for a year. To continue his research on theoretical physics, he came to the University of Florida in August, 1985.

Since then, he has shown broad interest in theoretical physics from superfluid ^3He to string theory and has concentrated on a geometric formulation of string field theory with emphasis on the geometry of loop space. Mathematical beauty of theoretical physics has always amazed him and deepened his understanding in physics, even though his earlier interest was on a tangiable structure of physics. The candidate expects to continue to search for the deep mathematical structure of physics for the next couple of years.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Pierre Ramond, Chairman
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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